

1. Some Additional Mathematics of the k -hat Exponential as a Rotation Matrix

The first document I wrote on this topic, k -cross and the k -hat exponential, showed that $i = k \times$ (k -cross) is the complex number for the xy -plane and then demonstrated the equivalence between the k -cross basis exponential **expk** and xy -plane rotation matrix by what I termed diadic expansion of the components of the operator. As an alternative to diadic expansion, presented here is another way to obtain the rotation matrix from the k -hat basis exponential **expk**, that is, the rotation matrix from what was formally known as the complex exponential (the exponential of $i = k \times$). This method makes use of showing the matrix basis components explicitly and operating the **unprimed left-to-right** k -hat basis exponential **expk** on the **unprimed basis vectors**, that is, on the column basis vectors in the matrix basis. This is very relevant to the theory of **Hermitian matrices** and potentially greatly increases the utility and proficient application of such theory.

The first line of maths is multiplying **expk** from the left with the identity I on the right (which is **expk** operating on I)

$$\exp(\theta \hat{k} \times) = e^{\theta \hat{k} \times} I = e^{\theta \hat{k} \times} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\theta \hat{k} \times} & 0 & 0 \\ 0 & e^{\theta \hat{k} \times} & 0 \\ 0 & 0 & e^{\theta \hat{k} \times} \end{pmatrix} = e^{\theta \hat{k} \times} \begin{pmatrix} 1 \hat{i} \hat{i}' & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & 1 \hat{j} \hat{j}' & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & 1 \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} e^{\theta \hat{k} \times} \hat{i} \hat{i}' & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & e^{\theta \hat{k} \times} \hat{j} \hat{j}' & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & e^{\theta \hat{k} \times} \hat{k} \hat{k}' \end{pmatrix}$$

from which **expk** then operates on the unprimed components in the matrix basis (of I). Substituting Euler's form **expk** $\theta = \text{cisk } \theta$ for the basis exponential, $e^{\theta \hat{k} \times} = \text{cos } \theta + (\text{sin } \theta) \hat{k} \times$, on the components on the identity's diagonal

$$\begin{aligned} \exp(\theta \hat{k} \times) &= \begin{pmatrix} (\text{cos } \theta + \text{sin } \theta \hat{k} \times) \hat{i} \hat{i}' & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & (\text{cos } \theta + \text{sin } \theta \hat{k} \times) \hat{j} \hat{j}' & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & (\text{cos } \theta + \text{sin } \theta \hat{k} \times) \hat{k} \hat{k}' \end{pmatrix} \\ &= \begin{pmatrix} (\text{cos } \theta \hat{i} + \text{sin } \theta \hat{k} \times \hat{i}) \hat{i}' & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & (\text{cos } \theta \hat{j} + \text{sin } \theta \hat{k} \times \hat{j}) \hat{j}' & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & (\text{cos } \theta \hat{k} + \text{sin } \theta \hat{k} \times \hat{k}) \hat{k}' \end{pmatrix} \\ &= \begin{pmatrix} (\hat{i} \cos \theta + \hat{j} \sin \theta) \hat{i}' & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & (\hat{j} \cos \theta - \hat{i} \sin \theta) \hat{j}' & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & (\hat{k} \cos \theta + \hat{i} \sin \theta) \hat{k}' \end{pmatrix} = \begin{pmatrix} (\hat{i} \hat{i}' \cos \theta + \hat{j} \hat{i}' \sin \theta) & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & (\hat{j} \hat{j}' \cos \theta - \hat{i} \hat{j}' \sin \theta) & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & (\hat{k} \hat{k}' \cos \theta + \hat{i} \hat{k}' \sin \theta) \end{pmatrix} \\ &= \begin{pmatrix} (\hat{i} \hat{i}' \cos \theta) & (0 - \hat{i} \hat{j}' \sin \theta) & 0 \hat{i} \hat{k}' \\ (0 + \hat{j} \hat{i}' \sin \theta) & (\hat{j} \hat{j}' \cos \theta) & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & (\hat{k} \hat{k}' \cos \theta + \hat{i} \hat{k}' \sin \theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta \hat{i} \hat{i}' & -\sin \theta \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ \sin \theta \hat{j} \hat{i}' & \cos \theta \hat{j} \hat{j}' & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & 1 \hat{k} \hat{k}' \end{pmatrix} \end{aligned}$$

which is the **basis exponential expk in rotation matrix form**. Here, noting that the diagonal components of the identity matrix, which are operated on by **expk**, are the components that are introduced in a diadic expansion by the components of **expk**. So this is essentially a more enlightening way of doing the same thing as diadic expansion. If it is just doing the same thing, then, then why bother? Because this approach can be varied to positions other than entries on the diagonal, specifically for the context of (this and other) **Hermitian matrices**. For example, if **expk** or $i = k \times$ or any of the other basis functions (such as **expj**, **expi**, $j \times$, or $i \times$) are located on entries other than the diagonal then this same approach is the approach that would be required. The shifting of $i = k \times$ (k -cross) around appropriately in the identity and expanded basis-exponential (Euler form) also demonstrates that rotation matrices are **Hermitian**. The definition of a **Hermitian matrix** is that the matrix is the transpose (T) of its conjugate ($\bar{\cdot}$). So the k -hat basis exponential **expk**, and equivalently the xy -plane rotation matrix, is **Hermitian** since

$$\exp(\theta \hat{k} \times) = \exp(-\theta \hat{k} \times)^T$$

that is, the $(k\text{-hat})$ basis exponential **expk** is the transpose of its own $(k\text{-cross})$ conjugate. In the previous complex number terminology, this is saying that the **complex exponential** is **Hermitian**, that is, after being diadically expanded as per the first $i = k \times$ document that I wrote.

The complex exponential is

$$e^{i\theta} = e^{i\theta} I = e^{i\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta + i\sin\theta & 0 & 0 \\ 0 & \cos\theta + i\sin\theta & 0 \\ 0 & 0 & \cos\theta + i\sin\theta \end{pmatrix}$$

and its conjugate is

$$e^{-i\theta} = e^{-i\theta} I = e^{-i\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta & 0 & 0 \\ 0 & \cos\theta - i\sin\theta & 0 \\ 0 & 0 & \cos\theta - i\sin\theta \end{pmatrix}$$

With $i = \hat{k} \times$ and cosine and sine rotating around $\hat{k}$ so that they have a $k\text{-hat}$ dependent base and so are $k\text{-hat}$ basis harmonics **cosk** and **sink**

$$e^{\theta \hat{k} \times} = e^{\theta \hat{k} \times} I = \begin{pmatrix} e^{\theta \hat{k} \times} & 0 & 0 \\ 0 & e^{\theta \hat{k} \times} & 0 \\ 0 & 0 & e^{\theta \hat{k} \times} \end{pmatrix} = \begin{pmatrix} \text{cosk}\theta + (\text{sink}\theta)\hat{k} \times & 0 & 0 \\ 0 & \text{cosk}\theta + (\text{sink}\theta)\hat{k} \times & 0 \\ 0 & 0 & \text{cosk}\theta + (\text{sink}\theta)\hat{k} \times \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the conjugate is

$$e^{-\theta \hat{k} \times} = e^{-\theta \hat{k} \times} I = \begin{pmatrix} e^{-\theta \hat{k} \times} & 0 & 0 \\ 0 & e^{-\theta \hat{k} \times} & 0 \\ 0 & 0 & e^{-\theta \hat{k} \times} \end{pmatrix} = \begin{pmatrix} \text{cosk}\theta - (\text{sink}\theta)\hat{k} \times & 0 & 0 \\ 0 & \text{cosk}\theta - (\text{sink}\theta)\hat{k} \times & 0 \\ 0 & 0 & \text{cosk}\theta - (\text{sink}\theta)\hat{k} \times \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the also the transpose. Hence, the complex exponential (basis exponential **expk**) is Hermitian. Similarly for the $j\text{-hat}$ and $i\text{-hat}$ basis exponentials **expj** and **expi**, that is, the $j\text{-hat}$ and $i\text{-hat}$ basis exponentials are also **Hermitian** since **expi** $\theta = \text{cisi}\theta$ and **expj** $\theta = \text{cisj}\theta$ are equal to the transposes of their conjugates. Again, this is now obvious since they are congruent and so are similar matrices once diadically expanded as shown by canonical forms and change of base between principal axes.

It may seem strange at first but $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is also Hermitian.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1\hat{i}\hat{i}' & 0\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ 0\hat{j}\hat{i}' & -1\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 1(\hat{k} \times \hat{j})\hat{i}' & 0\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ 0\hat{j}\hat{i}' & -1(\hat{k} \times \hat{i})\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & -(\hat{k} \times)\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ (\hat{k} \times)\hat{j}\hat{i}' & 1\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0 & -\hat{k} \times & 0 \\ \hat{k} \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and with $i = \hat{k} \times$ this matrix is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is a typical Hermitian matrix. Once you know about $i = \hat{k} \times$ it no longer seems strange since it is a complex number on a complex matrix (the $k \times$ curl positions) entry which produces a real result on the diagonal. This is the $k\text{-hat}$ double cross $\hat{k} \times \hat{k} \times$ which is Hermitian similarly as $i = \hat{k} \times$ and **expk**

$$\hat{k} \times \hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{which can be easily seen as} \quad \begin{pmatrix} 0 & -\hat{k} \times & 0 \\ \hat{k} \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{k} \times \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{k} \times \hat{k} \times$$

and why this matrix is periodic with period 2; it has only two states which are $(\hat{k} \times)^2 I$ and $(\hat{k} \times)^4 I$. The exponential of the double cross **expkk** is also Hermitian, as well as **expjj** and **expii** by change of base.

Theorem: all complex numbers (rotation operators $\hat{k} \times, \hat{j} \times, \hat{i} \times$) on an R^3 matrix basis diagonal are scalars **in their own curl (non-diagonal) positions** of the R^3 matrix basis and null for the rotation eigenvalue entry (once the complex number, as a rotation operator, has operated on the matrix basis).

For example,

$$\hat{k} \times I = \hat{k} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{k} \times & 0 & 0 \\ 0 & \hat{k} \times & 0 \\ 0 & 0 & \hat{k} \times \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with easy proof} \quad \hat{k} \times I = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and for an example with a bit more clarity

$$(a\hat{k} \times + b\hat{j} \times)I = (a\hat{k} \times + b\hat{j} \times) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a\hat{k} \times + b\hat{j} \times & 0 & 0 \\ 0 & a\hat{k} \times + b\hat{j} \times & 0 \\ 0 & 0 & a\hat{k} \times + b\hat{j} \times \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}$$

Converse: (1) complex numbers (rotation operators $\hat{k} \times, \hat{j} \times, \hat{i} \times$) on non-diagonals **in their own curl positions** in an R^3 matrix basis are scalars on the diagonal of the R^3 matrix basis (once the complex number, as a rotation operator, has operated on the matrix basis).

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{k} \times & 0 & 0 \\ 0 & \hat{k} \times & 0 \\ 0 & 0 & \hat{k} \times \end{pmatrix} = \begin{pmatrix} \hat{k} \times & 0 & 0 \\ 0 & \hat{k} \times & 0 \\ 0 & 0 & \hat{k} \times \end{pmatrix} = \hat{k} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hat{k} \times I = \hat{k} \times$$

and

$$\begin{pmatrix} 0 & -a & b \\ a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix} = \begin{pmatrix} a\hat{k} \times + b\hat{j} \times & 0 & 0 \\ 0 & a\hat{k} \times & 0 \\ 0 & 0 & b\hat{j} \times \end{pmatrix} = \begin{pmatrix} a\hat{k} \times + b\hat{j} \times & 0 & 0 \\ 0 & a\hat{k} \times + b\hat{j} \times & 0 \\ 0 & 0 & a\hat{k} \times + b\hat{j} \times \end{pmatrix} = (a\hat{k} \times + b\hat{j} \times) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (a\hat{k} \times + b\hat{j} \times)I = a\hat{k} \times + b\hat{j} \times$$

This also makes showing why $i = \hat{k} \times$ is Hermitian straight forward. The conjugate of $i = \hat{k} \times = \hat{k} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is

$$(\hat{k} \times)' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\hat{k} \times)' = \begin{pmatrix} (\hat{k} \times)' & 0 & 0 \\ 0 & (\hat{k} \times)' & 0 \\ 0 & 0 & (\hat{k} \times)' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so is also the transpose of $i = \hat{k} \times$ and is therefore Hermitian. Clearer for those still thinking in terms of complex i

$$i = iI = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} = \hat{k} \times = \hat{k} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{k} \times & 0 & 0 \\ 0 & \hat{k} \times & 0 \\ 0 & 0 & \hat{k} \times \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and, not complicating things with reversal of direction of the cross product triangle (just taking the negative of $\hat{k} \times$ instead) its conjugate is

$$-i = -iI = \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} = -\hat{k} \times = -\hat{k} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\hat{k} \times & 0 & 0 \\ 0 & -\hat{k} \times & 0 \\ 0 & 0 & -\hat{k} \times \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is its transpose and so is Hermitian similarly as all functions of $\hat{k} \times$.

Theorem: $\hat{k} \times$ has complex number properties for the $\hat{k} \times$ curl positions whilst nilpotent properties for the $\hat{j} \times$ and $\hat{i} \times$ curl positions in the R^3 matrix basis. Similarly, $\hat{j} \times$ has complex number properties for the $\hat{j} \times$ curl positions whilst nilpotent properties for the $\hat{k} \times$ and $\hat{i} \times$ curl positions in the R^3 matrix basis. Similarly, $\hat{i} \times$ has complex number properties for the $\hat{i} \times$ curl positions whilst nilpotent properties for the $\hat{j} \times$ and $\hat{k} \times$ curl positions in the R^3 matrix basis.

Theorem: moving entries from the non-diagonal entries of an R^3 matrix basis is a basis algebra technique of diagonalizing an R^3 matrix basis. Whichever entries on the diagonal are homogenous (in $\hat{k} \times$, $\hat{j} \times$, or $\hat{i} \times$) can be raised to powers and inverted with standard complex algebra (nb: because these operators are singular, they need to be multiplied by cubes as proxy to invert).

Rather than $\hat{k} \times$ in the k -cross positions, which results in the k -hat double cross $\hat{k} \times \hat{k} \times$, if the complex number had have been either j -cross or i -cross in the k -cross positions then the resulting matrix is a matrix with a single nilpotent entry. These are the nilpotent mixed-double-crosses. In the example with i in the k -cross positions $\hat{j} \times \hat{k} \times$ or $\hat{j} \times \hat{k} \times$.

Theorem: complex numbers (rotation operators $\hat{k} \times$, $\hat{j} \times$, $\hat{i} \times$) on non-diagonals positions that are **not in their own curl positions** in an R^3 matrix basis are either zero or nilpotent scalar on other non-diagonal positions not corresponding to **their own non-diagonal curl positions**.

Considering again the matrix with complex number i in the k -cross positions

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If j -cross is taken as the complex number here, $i = \hat{j} \times$, the result will be a nilpotent matrix because the entries are located in the k -cross curl positions rather than the j -cross curl positions

$$\begin{pmatrix} 0 & -\hat{j} \times & 0 \\ \hat{j} \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & -(\hat{j} \times)\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ (\hat{j} \times)\hat{j}\hat{i}' & 1\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & -1(\hat{j} \times \hat{i})\hat{j}' & 0\hat{i}\hat{k}' \\ 1(\hat{j} \times \hat{j})\hat{i}' & 1\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & 0\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ (0\hat{j})\hat{i}' & 0\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & (-1\hat{k})\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

since $(\hat{j} \times \hat{j})\hat{i}' = (0\hat{j})\hat{i}' = 0\hat{j}\hat{i}'$ and $(\hat{j} \times \hat{i})\hat{j}' = (-1\hat{k})\hat{j}' = -1\hat{k}\hat{j}'$. This is the nilpotent mixed double cross, j -cross- k -cross ($\hat{j} \times \hat{k} \times$), since it is j -cross in the k -cross position

$$\begin{pmatrix} 0 & -\hat{j} \times & 0 \\ \hat{j} \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{j} \times \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{j} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Similarly if i -cross is taken as the complex number, $i = \hat{i} \times$, the result will be a nilpotent matrix similarly because the entries are located in the k -cross curl positions rather than the i -cross curl positions

$$\begin{pmatrix} 0 & -\hat{i} \times & 0 \\ \hat{i} \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & -(\hat{i} \times)\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ (\hat{i} \times)\hat{j}\hat{i}' & 1\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & -1(\hat{i} \times \hat{i})\hat{j}' & 0\hat{i}\hat{k}' \\ 1(\hat{i} \times \hat{j})\hat{i}' & 1\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & (0\hat{i})\hat{j}' & 0\hat{i}\hat{k}' \\ 0\hat{j}\hat{i}' & 0\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ (+1\hat{k})\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

since $(\hat{i} \times \hat{j})\hat{i}' = (+1\hat{k})\hat{i}' = 1\hat{k}\hat{i}'$ and $(\hat{i} \times \hat{i})\hat{j}' = (0\hat{i})\hat{j}' = 0\hat{i}\hat{j}'$. This is the lower triangular squared nilpotent matrix and the nilpotent mixed double cross, i -cross- k -cross ($\hat{i} \times \hat{k} \times$), because it is i -cross in the k -cross position

$$\begin{pmatrix} 0 & -\hat{i} \times & 0 \\ \hat{i} \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{i} \times \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{i} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Therefore discretion is required to choose correctly the complex number required for one's purpose. If it is to have periodic properties, it must be located on the diagonal or in its own curl positions otherwise the periodic property will be lost to nilpotence. Nilpotence in some circumstances is the required result.

There is also the issue of whether the rotation operators are primed or unprimed, and hence **left-to-right operators** or **right-to-left operators** operating on, respectively, the **unprimed column vector basis** or the **primed row vector basis** in the **matrix basis**. If

primed k -cross is taken as the complex number, then the matrix will lose its periodic properties, since it is the product of k -cross with its conjugate, so is identity with the singular position zeroed (because k -cross is singular)

$$\begin{pmatrix} 0 & -(\hat{k} \times)' & 0 \\ (\hat{k} \times)' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\hat{k} \times)' = \hat{k} \times (\hat{k} \times)' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is singular idempotent. In full matrix basis algebra

$$\begin{pmatrix} 0 & -(\hat{k} \times)' & 0 \\ (\hat{k} \times)' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & -1\hat{i}\hat{j}'(\hat{k} \times)' & 0\hat{i}\hat{k}' \\ 1\hat{j}\hat{i}'(\hat{k} \times)' & 0\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 0\hat{i}\hat{i}' & -\hat{i}(\hat{j}' \times \hat{k}') & 0\hat{i}\hat{k}' \\ \hat{j}(\hat{i}' \times \hat{k}') & 1\hat{j}\hat{j}' & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} -\hat{i}(-1\hat{i}') & 0\hat{i}\hat{j}' & 0\hat{i}\hat{k}' \\ 0\hat{j}\hat{i}' & \hat{j}(+1\hat{j}') & 0\hat{j}\hat{k}' \\ 0\hat{k}\hat{i}' & 0\hat{k}\hat{j}' & 0\hat{k}\hat{k}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\hat{i}(\hat{j}' \times \hat{k}') = \hat{i}(-\hat{i}') = -1\hat{i}\hat{i}'$ and $\hat{j}(\hat{i}' \times \hat{k}') = \hat{j}(+1\hat{j}') = +1\hat{j}\hat{j}'$ noting the reversal of the directionality of the cross product triangle which is discussed in the next topic.

If either primed j -cross or primed i -cross are taken as the complex numbers, then the resulting matrix operator will be the same nilpotent matrices as above but will be upper triangular (rather than lower triangular)

$$\begin{pmatrix} 0 & -(\hat{j} \times)' & 0 \\ (\hat{j} \times)' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -(\hat{i} \times)' & 0 \\ (\hat{i} \times)' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So discretion is required at determining which complex number one is working with although it all works out nicely if the right one is chosen. k -cross in the k -cross position is the most sensible standard in my opinion but whatever works best for the situation. Jordan matrices, for example, require nilpotent operators.

Another interesting periodic matrix that I have noticed in times past while studying matrix mechanics quantum (the second half of a standard 2nd year University course in Quantum Mechanics) was the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which has the same form as $\hat{k} \times$ except without the negative in either of the two $\hat{k} \times$ curl positions. This matrix is equivalent to $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$. This is a hyperbolic analogy to $\hat{k} \times$ which has circular complex number identity $\hat{k} \times = \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$. The complete set are

$$\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

which are hyperbolic forms that correspond to the harmonic forms of $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$

$$\hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times = \hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{k} \times \hat{i} \times - \hat{i} \times \hat{k} \times = \hat{j} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \hat{j} \times \hat{k} \times - \hat{k} \times \hat{j} \times = \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

noting $\hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times = \hat{k} \times$, $\hat{k} \times \hat{i} \times - \hat{i} \times \hat{k} \times = \hat{j} \times$, and $\hat{j} \times \hat{k} \times - \hat{k} \times \hat{j} \times = \hat{i} \times$ are basis operator identities for $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$.

The exponentials of $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ (complex exponentials) with argument θ diadically expand into the rotation matrices **expk** θ , **expj** θ , and **expi** θ . Interestingly, the exponentials of $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$, $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$, and $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ diadically expand into those same matrices but with hyperbolics cosh and sinh instead of harmonics cos and sin, and further, still permute over the respective axis unlike the double crosses (which are then attenuation factors on the diagonal... these here, in contrast to the double crosses, remain distributed on the curl positions which is why they are interesting). I will demonstrate it for $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ and then change base to those for $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$ and $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$. I am thinking of calling these **expkh**, **expjh**, and **expih** (for hyperbolic exponentials analogous to the single cross exponentials **expk**, **expj**, and **expi** and the double cross exponentials **expkk**, **expjj**, and **expii**).

Exponentiating $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ and expanding

$$\exp(\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) = +\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^2 + \frac{1}{3!} \theta^3 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^3 \\ + \frac{1}{4!} \theta^4 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^4 + \frac{1}{5!} \theta^5 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^5 + \frac{1}{6!} \theta^6 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^6 + \dots$$

i-hat column basis component

$$\exp(\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) \hat{i} = \hat{i} + \theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \hat{i} + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^2 \hat{i} + \frac{1}{3!} \theta^3 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^3 \hat{i} \\ + \frac{1}{4!} \theta^4 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^4 \hat{i} + \frac{1}{5!} \theta^5 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^5 \hat{i} + \frac{1}{6!} \theta^6 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^6 \hat{i} + \dots \\ = \hat{i} + \theta \hat{j} + \frac{1}{2!} \theta^2 \hat{i} + \frac{1}{3!} \theta^3 \hat{j} + \frac{1}{4!} \theta^4 \hat{i} + \frac{1}{5!} \theta^5 \hat{j} + \frac{1}{6!} \theta^6 \hat{i} + \dots \\ = \hat{i} \left(1 + \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 + \frac{1}{6!} \theta^6 + \dots\right) + \hat{j} \left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots\right) \\ = \hat{i} \cosh \theta + \hat{j} \sinh \theta$$

$$\text{since } (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^n \hat{i} = \begin{cases} \hat{i} & \text{when } n \text{ is even} \\ \hat{j} & \text{when } n \text{ is odd} \end{cases}$$

j-hat column basis component

$$\exp(\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) \hat{j} = \hat{j} + \theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \hat{j} + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^2 \hat{j} + \frac{1}{3!} \theta^3 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^3 \hat{j} \\ + \frac{1}{4!} \theta^4 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^4 \hat{j} + \frac{1}{5!} \theta^5 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^5 \hat{j} + \frac{1}{6!} \theta^6 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^6 \hat{j} + \dots \\ = \hat{j} + \theta \hat{i} + \frac{1}{2!} \theta^2 \hat{j} + \frac{1}{3!} \theta^3 \hat{i} + \frac{1}{4!} \theta^4 \hat{j} + \frac{1}{5!} \theta^5 \hat{i} + \frac{1}{6!} \theta^6 \hat{j} + \dots \\ = \hat{j} \left(1 + \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 + \frac{1}{6!} \theta^6 + \dots\right) + \hat{i} \left(\theta + \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \dots\right) \\ = \hat{i} \sinh \theta + \hat{j} \cosh \theta$$

$$\text{since } (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^n \hat{j} = \begin{cases} \hat{j} & \text{when } n \text{ is even} \\ \hat{i} & \text{when } n \text{ is odd} \end{cases}$$

k-hat column basis component

$$\exp(\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) \hat{k} = \hat{k} + \theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \hat{k} + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^2 \hat{k} + \frac{1}{3!} \theta^3 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^3 \hat{k} \\ + \frac{1}{4!} \theta^4 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^4 \hat{k} + \frac{1}{5!} \theta^5 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^5 \hat{k} + \frac{1}{6!} \theta^6 (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^6 \hat{k} + \dots \\ = \hat{k} + 0\hat{k} + 0\hat{k} + 0\hat{k} + \dots \\ = \hat{k}$$

$$\text{since } (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)^n \hat{k} = \begin{cases} \hat{k} & n = 0 \\ 0 & n > 0 \end{cases}$$

The very interesting thing here is that the *i*-hat and *j*-hat components of **exphk** osculate between *i*-hat and *j*-hat axes which differs from the *k*-hat double cross **expkk** which osculates in sign on the same *i*-hat or *j*-hat axis (corresponding to the component axis). This means that these basis hyperbolics (**exphk**, **exphj**, and **exphi**) are definitely different from the double cross hyperbolics (**expkk**, **expjj**, and **expii**) and, to my credit, why I didn't call the double crosses hyperbolics when I first found them (and named them **expkk** rather than **exph**, although had considered it, which is turning out to be the right thing to do). The components then diadically expand (components to columns) into

$$\mathbf{exphk} \theta = \exp(\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) = \begin{pmatrix} \cosh \theta & \sinh \theta & 0 \\ \sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Unlike the *k*-hat double cross exponential **expkk** with hyperbolics in the **coskk** and **sinkk**, this matrix **exphk** has a unitary determinant and is also symmetric in that it is equal to its own transpose. This is by virtue of the components osculating between axes (with period two rather than four for harmonics) rather than osculating in sign on the same axis as for the double crosses.

The basis hyperbolics are, for hyperbolic cosine, **coshk**

$$\mathbf{coshk} \theta = \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{suggested pronunciation "cosh-kay-theta"})$$

and for hyperbolic sines, **sinhk**

$$\mathbf{sinhk}\theta = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \quad (\text{suggested pronunciation "shine-kay-theta"})$$

although the k -hat dependence in the naming is slightly ambiguous because these basis exponentials do not have a rotation axis so the complex number does not involve k -hat, instead it only involves the basis vectors that generate the orthogonal complement to k -hat, the xy plane generated by i -hat and j -hat.

Also, similarly as the other basis cosines, $\mathbf{coshk}$ is full rank so can be divided into (inverted and multiplied with) $\mathbf{sinhk}$ to define $\mathbf{tanhk}$, hyperbolic tangent for the xy plane (with suggested pronunciation "than-kay")

$$\mathbf{tanhk}\theta = (\mathbf{sinhk}\theta)(\mathbf{coshk}\theta)^{-1} = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} \operatorname{sech} \theta & 0 & 0 \\ 0 & \operatorname{sech} \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{tanhk}\theta = \begin{pmatrix} \tanh \theta & 0 & 0 \\ 0 & \tanh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

These are actually the same as $\mathbf{coskk}$, $\mathbf{sinkk}$ (with a sign change), and $\mathbf{tankek}$ but differ when they combine into $\mathbf{exphk}$ (that is, differ from $\mathbf{expkk}$) because of the osculating properties, that is, because they **axis osculate** rather than **sign osculate**.

With $\mathbf{coshk}$ and $\mathbf{sinhk}$, the Euler form of $\mathbf{exphk}$ by diad decomposition to even and odd components is therefore

$$\mathbf{exphk}\theta = \mathbf{coshk}\theta + (\mathbf{sinhk}\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$$

in other words (diadically expanded matrix base),

$$\begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh \theta & \sinh \theta & 0 \\ \sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has de-Moivre powers

$$\begin{aligned} (\mathbf{exphk}\theta)^n &= \mathbf{exphk}(n\theta) \\ &= \mathbf{coshk}(n\theta) + (\mathbf{sinhk}(n\theta))(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) \end{aligned}$$

i.e.,

$$(\mathbf{exphk}\theta)^n = \begin{pmatrix} \cosh n\theta & \sinh n\theta & 0 \\ \sinh n\theta & \cosh n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} \cosh n\theta & 0 & 0 \\ 0 & \cosh n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sinh n\theta & 0 & 0 \\ 0 & \sinh n\theta & 0 \\ 0 & 0 & n\theta \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh n\theta & \sinh n\theta & 0 \\ \sinh n\theta & \cosh n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The conjugate of the hyperbolic complex number $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ is either that with the sign changed $-\hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times$ or with a change of base to $\hat{k} \times (\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)(\hat{k} \times)'$ which diadically expands (or alternatively changing the sign of the diadically expanded form of $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$) to

$$-(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which has basis exponential

$$\mathbf{exphk}\theta = \exp(-\theta(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 \\ -\sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the Hermite conjugate of **exphk** (which considering $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ as the hyperbolic complex number), and conjugate Euler form

$$\mathbf{exphk}\theta = \cosh\theta - (\sinh\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)$$

By change of base of **exphk** with the appropriate quarter axis turn basis exponentials **expi** and **expj** to the yz-plane and the zx-plane, respectively, then results in the basis hyperbolics **expjh** and **expjh**. For **expjh**, the change of base matrix is **exp**($\pi/2$).

Applying the change of base to $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ results in the hyperbolic complex number $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$ for the j -hat orthogonal complement [the plane generated and spanned by i -hat and k -hat]

$$\mathbf{expi}(\pi/2)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)\mathbf{expi}(-\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$$

and so all of the basis functions that depend on $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ will have their based changed to that of $\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times$ by this change of base with **expi** (from the **change of base theorem** that a change of a base of a function of a base equals that function with its base changed by that change of base [in the change of principal axis document that I wrote]). For **expjh**

$$\mathbf{expi}(\pi/2)(\mathbf{exphk}\theta)\mathbf{expi}(-\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

Therefore the j -hat hyperbolic exponential **expjh** is

$$\mathbf{expjh}\theta = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

and changing the base of the basis hyperbolics **coshk** and **sinhk** to those of **coshj** and **sinhj**

$$\mathbf{expi}(\pi/2)(\mathbf{coshk}\theta)\mathbf{expi}(-\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix}$$

$$\mathbf{expi}(\pi/2)(\mathbf{sinhk}\theta)\mathbf{expi}(-\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix}$$

that is,

$$\mathbf{coshj}\theta = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} \quad \text{and} \quad \mathbf{sinhj}\theta = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix}$$

The Euler form diad decomposition of **expjh** is then

$$\mathbf{expjh}\theta = \mathbf{coshj}\theta + (\mathbf{sinhj}\theta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times)$$

i.e.,

$$\begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} + \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with de-Moivre powers

$$\begin{aligned} (\mathbf{expjh}\theta)^n &= \mathbf{expjh}(n\theta) \\ &= \mathbf{coshj}(n\theta) + (\mathbf{sinhj}(n\theta))(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) \end{aligned}$$

Making a change of base of **exphk** to the i -hat hyperbolic **exphi**, firstly by changing the hyperbolic complex number $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ of the plane of the base with **expj** $(-\pi/2)$ to that of $\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times$

$$\mathbf{expj}(-\pi/2)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times)\mathbf{expj}(-\pi/2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times$$

and so by the function of changes of base theorem, all of the functions for $\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times$ will be converted to those in $\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times$ with this similarity transform to the orthogonal plane spanned by j -hat and k -hat. Changing the base of **exphk** to that of **exphi**

$$\mathbf{expj}(-\pi/2)(\mathbf{exphk} \theta)\mathbf{expj}(-\pi/2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \theta & \sinh \theta & 0 \\ \sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix}$$

and therefore the hyperbolic basis exponential **exphi** is

$$\mathbf{exphi} \theta = \exp(\theta(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix}$$

The hyperbolics for the plane spanned by j -hat and k -hat **coshi** and **sinhi** (and hence also **tanhi**) are from change of base with **expj**

$$\mathbf{expj}(-\pi/2)(\mathbf{coshk} \theta)\mathbf{expj}(\pi/2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh \theta & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix}$$

$$\mathbf{expj}(-\pi/2)(\mathbf{sinhk} \theta)\mathbf{expj}(\pi/2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sinh \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \sinh \theta \end{pmatrix}$$

that is,

$$\mathbf{coshi} \theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix}, \quad \mathbf{sinhi} \theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \sinh \theta \end{pmatrix}, \quad \text{and} \quad \mathbf{tanhi} \theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \tanh \theta & 0 \\ 0 & 0 & \tanh \theta \end{pmatrix}$$

The Euler form of **exphi** is

$$\mathbf{exphi} \theta = \mathbf{coshi} \theta + (\mathbf{sinhi} \theta)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times)$$

i.e.,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & 0 \\ 0 & 0 & \cosh \theta \end{pmatrix} + \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh \theta & 0 \\ 0 & 0 & \sinh \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with de-Moivre powers

$$\begin{aligned} (\mathbf{exphi} \theta)^n &= \mathbf{exphi} (n\theta) \\ &= \mathbf{coshi}(n\theta) + (\mathbf{sinhi}(n\theta))(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) \end{aligned}$$

Note that I have $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ programmed in my TI-89 as $k\mathbf{c}$ (for k -cross), $j\mathbf{c}$ (for j -cross), and $i\mathbf{c}$ (for i -cross), and i (for i -hat), j (for j -hat), and k (for k -hat) and so check all the mathematics easily and I highly recommend that someone that is interested in this mathematics does so also (noting that i , j , and k are 3by1 column vectors whilst $i\mathbf{c}$, $j\mathbf{c}$, and $k\mathbf{c}$ are 3by3 matrices) with

$$\hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{j} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and try firstly $ic*jc+jc*ic$ and then raise this to powers $(ic*jc+jc*ic)^n$ demonstrating the order 2 periodicity for $n = 1, 2, 3, 4, \dots$. Then exponentiate $ic*jc+jc*ic$ and observe that it is **expjk**(1) with two diagonal entries $\cosh(1)=1.543$ and the two anti-diagonals $\sinh(1)=1.175$ in the $k \times$ positions. And then change the sign of the hyperbolic complex number converting it to the harmonic complex number $ic*jc-jc*ic$ and then exponentiate that and observe that it is the rotation matrix **expk**(1) [argument in radians] with two diagonal entries $\cos(1)=0.540$ and two anti-diagonal entries $\sin(1)=0.841$ with that in the first row having a negative sign (the k -cross negative). Multiplying (with $*$) any of these by i (i -hat), j (j -hat), or k (k -hat) then gives the column vector of that component since these, **expk** and **expjk** for instance, are operators for position vectors (just a reminder).

This is how I have this mathematics programmed into my TI at the moment, in radians with pretty print on and graphing by parametric form since all of these functions (other than i -hat, j -hat, and k -hat, and i -cross, j -cross, and k -cross) are parameterized by θ which is ordinarily ωt . This is the exact syntax in my TI

BASIS VECTORS

$$i = [1,0,0]^T$$

$$j = [0,1,0]^T$$

$$k = [0,0,1]^T$$

All of the functions, other than the basis vectors themselves, are operators so that the cross is then part of the defined right-to-left functions. The ordinary multiply operator $*$ is then, for all intents and purposes, the cross operator.

SINGULAR COMPLEX NUMBERS (BASIS VECTORS APPENDED WITH CROSS)

$$ic = [[0,0,0][0,0,-1][0,1,0]]$$

$$jc = [[0,0,1][0,0,0][-1,0,0]]$$

$$kc = [[0,-1,0][1,0,0][0,0,0]]$$

BASIS EXPONENTIALS (HARMONIC, ROTATION)

$$\text{define } ei(\theta) = [[\cos(\theta), -\sin(\theta), 0][\sin(\theta), \cos(\theta), 0][0, 0, 1]]$$

$$\text{define } ej(\theta) = [[\cos(\theta), 0, \sin(\theta)][0, 0, 0][-\sin(\theta), 0, \cos(\theta)]]$$

$$\text{define } ek(\theta) = [[1, 0, 0][0, \cos(\theta), -\sin(\theta)][0, \sin(\theta), \cos(\theta)]]$$

BASIS EXPONENTIALS (DOUBLE CROSS ATTENUATION)

$$\text{define } eii(\theta) = [[\exp(-\theta), 0, 0][0, \exp(-\theta), 0][0, 0, 1]]$$

$$\text{define } ejj(\theta) = [[\exp(-\theta), 0, 0][0, 1, 0][0, 0, \exp(-\theta)]]$$

$$\text{define } ekk(\theta) = [[1, 0, 0][0, \exp(-\theta), 0][0, 0, \exp(-\theta)]]$$

BASIS EXPONENTIALS (HYPERBOLIC): add h to the harmonics **expk**, **expj**, and **expi** and remove the negative sign

$$\text{define } ehi(\theta) = [[\cosh(\theta), \sinh(\theta), 0][\sinh(\theta), \cosh(\theta), 0][0, 0, 1]]$$

$$\text{define } ehj(\theta) = [[\cosh(\theta), 0, \sinh(\theta)][0, 0, 0][\sinh(\theta), 0, \cosh(\theta)]]$$

$$\text{define } ehk(\theta) = [[1, 0, 0][0, \cosh(\theta), \sinh(\theta)][0, \sinh(\theta), \cosh(\theta)]]$$

FULL RANK COMPLEX NUMBER: change of base matrix from k -hat rotation basis to j -hat rotation basis $ei(\pi/2)$

FULL RANK COMPLEX NUMBER: change of base matrix from k -hat rotation basis to i -hat rotation basis $ej(-\pi/2)$

These are also the change of base matrices to cylindrical and spherical coordinates by cis scaling the standard rectangular basis.

BASIS COSINES (HARMONIC, ROTATION)

$$\text{define } ci(\theta) = [[\cos(\theta), 0, 0][0, \cos(\theta), 0][0, 0, 1]]$$

$$\text{alternatively, define } ci(\theta) = (ei(\theta) + ei(-\theta))/2$$

$$\text{define } cj(\theta) = [[\cos(\theta), 0, 0][0, 1, 0][0, 0, \cos(\theta)]]$$

$$\text{alternatively, define } cj(\theta) = (ej(\theta) + ej(-\theta))/2$$

$$\text{define } ck(\theta) = [[1, 0, 0][0, \cos(\theta), 0][0, 0, \cos(\theta)]]$$

$$\text{alternatively, define } ck(\theta) = (ek(\theta) + ek(-\theta))/2$$

BASIS COSINES (DOUBLE CROSS ATTENUATION)

define cii(θ) = [[cosh(θ),0,0][0,cosh(θ),0][0,0,1]]

alternatively, define chi(θ) = (eii(θ)+eii(-θ))/2

define cij(θ) = [[cosh(θ),0,][0,1,0][0,0,cosh(θ)]]

alternatively, define cjj(θ) = (ejj(θ)+ejj(-θ))/2

define ckk(θ) = [[1,0,0][0,cosh(θ),0][0,0,cosh(θ)]]

alternatively, define ckk(θ) = (ekk(θ)+ekk(-θ))/2

BASIS COSINES (HYPERBOLIC)

define cih(θ) = [[cosh(θ),0,0][0,cosh(θ),0][0,0,1]]

alternatively, define chi(θ) = (ehi(θ)+ehi(-θ))/2

define chj(θ) = [[cosh(θ),0,][0,1,0][0,0,cosh(θ)]]

alternatively, define chj(θ) = (ehj(θ)+ehj(-θ))/2

define chk(θ) = [[1,0,0][0,cosh(θ),0][0,0,cosh(θ)]]

alternatively, define chk(θ) = (ehk(θ)+ehk(-θ))/2

BASIS SINES (HARMONIC, ROTATION)

define si(θ) = [[sin(θ),0,0][0,sin(θ),0][0,0,θ]]

define sj(θ) = [[sin(θ),0,0][0,θ,0][0,0,sin(θ)]]

define sk(θ) = [[θ,0,0][0,sin(θ),0][0,0,sin(θ)]]

BASIS SINES (DOUBLE CROSS ATTENUATION)

define sii(θ) = [[exp(-θ),0,0][0,exp(-θ),0][0,0,1]]

define sji(θ) = [[exp(-θ),0,0][0,1,0][0,0,exp(-θ)]]

define skk(θ) = [[1,0,0][0,exp(-θ),0][0,0,exp(-θ)]]

BASIS SINES (HYPERBOLIC)

define shi(θ) = [[sinh(θ),0,0][0,sinh(θ),0][0,0,θ]]

define shj(θ) = [[sinh(θ),0,0][0,θ,0][0,0,sinh(θ)]]

define ehk(θ)= [[1,0,0][0,cosh(θ),sinh(θ)][0,sinh(θ),cosh(θ)]]

It works like a dream. Overall, this will allow us to do much more complicated mathematics than previously and much more intuitively. This is so since all the steps in how it works for R^3 are clear as well as consistent with the axioms of linear algebra.

2. Further Mathematics of the $\hat{k}$ -hat Exponential Conjugate $\exp k'$: reversal of direction of the cross-product triangles for primed basis variables and primed cross-products

Here I show that the conjugate $\hat{k}$ -hat exponential ($\exp k$)' is in the literal sense the exponential of the conjugate of $\hat{k}$ -cross ($\hat{k} \times$)', that is, the exponential of primed $\hat{k}$ -cross ($\hat{k} \times$)' = $\times \hat{k}'$. This is an explicit priming of $\hat{k} \times$ rather than just a sign change for the conjugate in the pair. This demonstrates that the change in the direction of the rotation matrix is more than just the sign changes on the sines in the $\hat{k} \times$ positions [the ij' and ji' positions in the R^3 matrix basis]. It reverses the direction of the cross product triangle which explains the change of sign.

This also demonstrates the equivalence between **conjugates in complex number pairs** and **Hermitian conjugates in linear algebra**. These are the same thing, that is, once a complex number is recognised as being a basis vector appended with $\times$ and subsequently **diadically expanded** into **matrix operator form**. This matrix form is **Hermitian**.

In ordinary scalar mathematics, one does not otherwise have to be concerned with the reason behind the sign change.

Theorem: $\hat{k} \times$, $\hat{j} \times$, $\hat{i} \times$, $\exp k$, $\exp j$, and $\exp i$ are Hermitian. This is shown, as per the previous section by moving the terms in either $\hat{k} \times$, $\hat{j} \times$, or $\hat{i} \times$ onto the main diagonal, and changing their sign for the Hermitian conjugate.

For an unprimed basis-cross operator operating **left-to-right** on an unprimed basis component, here

i -cross operating **left-to-right** on k -hat

and

 j -cross operating **left-to-right** on k -hat

$$(\hat{i} \times) \hat{k} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\hat{j}$$

$$(\hat{j} \times) \hat{k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hat{i}$$

which is (for i -cross operating **left-to-right** on k -hat) negative j -hat and the **usual clockwise (cw) direction** of the usual **cross-product triangle** (similarly for j -cross operating **left-to-right** on k -hat). Whilst on the other hand, a primed basis-hat being operated on from the right by a primed basis-cross operator, here primed i -hat being operated on from the **right (to the left)** by primed k -cross (cross- k -prime) and primed j -hat being operated on from the **right (to left)** by primed k -cross

$$\hat{i}'(\times \hat{k}') = (1, 0, 0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0, 1, 0) = \hat{j}'$$

$$\hat{j}'(\times \hat{k}') = (0, 1, 0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (-1, 0, 0) = -\hat{i}'$$

which is positive j -hat-prime and the opposite (**counter-clockwise direction**) cyclic direction (in terms of sign) of the usual **cross-product triangle**, if read **left-to-right** since it is positive, or depending on how one looks at it, the same cyclic direction as the usual cross product triangle if the maths is read from **right-to-left**... from **primed operator [conjugate complex number]** on the **right to primed operand [primed basis component]** on the **left**. This is another item of maths which is slightly disconcerting [and frustrating at first], and at first seems trivial even pedantic [pulling ones hair out in frustration over a sign], but nevertheless I have found important to get straight so that it all works as it should. This is the reason why the vector triple product is not associative. Operating on primed j -hat from the right-to-the-left by the primed k -hat exponential

$$\begin{aligned} \hat{j}' \exp(\times \hat{k}' \theta) &= \hat{j}' + \hat{j}'(\times \hat{k}')\theta + \frac{1}{2!} \hat{j}'(\times \hat{k}')^2 \theta^2 + \frac{1}{3!} \hat{j}'(\times \hat{k}')^3 \theta^3 + \dots \\ &= \hat{j}' - \hat{i}'\theta - \frac{1}{2!} \hat{j}'\theta^2 + \frac{1}{3!} \hat{i}'\theta^3 + \dots \\ &= -\left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right) \hat{i}' + \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots\right) \hat{j}' \\ &= (-\sin \theta) \hat{i}' + (\cos \theta) \hat{j}' \end{aligned}$$

The j -hat direction for the k -hat exponential is negative for the sine term. At first I thought it would be the opposite since I was attempting to map components to columns, but these terms **map to rows** (rather than columns) of the subsequent diadic expansion into the conjugate basis exponential. This is obvious now in hindsight. This is therefore the second row (the $\hat{j}$ row) in the matrix basis, rather than the second column. This demonstrates why it is necessary to describe the directions of the operators and cross product triangles with respect to priming in fine detail [verging on pedantism]. In doing so, this has also addressed the fact of the **non-associative vector triple product**. The theory of priming basis vectors appended with $\times$, and therefore operator direction, therefore explains this.

The **vector triple product** is two consecutive sets of basis crosses (quaternion complex numbers) operating on a basis vector at the same primed order. So for example

$$(\hat{k} \times)(\hat{k} \times) \hat{i} = (\hat{k} \times) \hat{j} = -\hat{i} \quad \text{differs from} \quad \hat{k}'(\times \hat{k}')(\times \hat{i}') = 0 \hat{k}'(\times \hat{i}') = 0(-\hat{j}') = 0 \hat{j}'$$

the difference for which makes clear that precedence is essential because of the reversed order of the cross product triangle upon priming and the resulting differing sequential order of precedence between **left-to-right operators (unprimed)** and **right-to-left operators (primed)** applied to the operand. This, then, is **primed and unprimed** complex numbers and quaternions. The first of these two comparison examples $(\hat{k} \times)(\hat{k} \times) \hat{i} = (\hat{k} \times) \hat{j} = -\hat{i}$ is a double cross (i -squared operating **left-to-right** equals negative one, -1) and the second $\hat{k}'(\times \hat{k}')(\times \hat{i}') = 0 \hat{k}'(\times \hat{i}') = 0(-\hat{j}') = 0 \hat{j}'$ is a primed nilpotent operator (N_3^2 operating **right-to-left** on k -hat which is zero, 0):

$$(\hat{k} \times)(\hat{k} \times) \hat{i} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (\hat{k} \times)^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -\hat{i}$$

and

$$\hat{k}'(\times \hat{k}')(\times \hat{i}') = (0, 0, 1) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = (0, 0, 1) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0, 0, 1) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = (0, 0, 1) N_3^2 = (0, 0, 0) = 0 \hat{k}'$$

So this **operator bracketing notation** of $\hat{k} \times$, here for $(\hat{k} \times)(\hat{k} \times)\hat{i} = (\hat{k} \times)\hat{j} = -\hat{i}$ and $\hat{k}'(\times \hat{k}')(\times \hat{i}') = 0\hat{k}'(\times \hat{i}') = 0(-\hat{j}') = 0\hat{j}'$ is more effective mathematical notation, in this context, than **associative bracketing notation** $\hat{k} \times (\hat{k} \times \hat{i})$ and $(\hat{k}' \times \hat{k}') \times \hat{i}'$ for vector triple product since it shows why the results are different by reference to directions of cross-product triangles.

Theorem: The **cross-product-triangle** of the **primed basis vectors** reverses the direction of the **standard unprimed cross-product-triangle** – this is then (1) **negative** for the **counter-clockwise direction** for **unprimed (standard) basis vectors** and (2) **negative** for the **clockwise direction** for **primed basis vectors**.

Corollary: This amounts to a sign change for the conjugate of the basis cross and so conjugate of the complex number like properties in complex conjugate and Hermite conjugate pairs.

Theorem: $k \times$ and $\mathbf{exp}k$ are transpose dependent. Similarly also are $j \times$, $i \times$, $\mathbf{exp}j$, and $\mathbf{exp}i$ by similarity and congruence. In scalar mathematics, the transpose of $k \times$ corresponds to the sign change of the conjugate.

Theorem (transpose independence of basis harmonics): the k -hat basis sines and cosines, **sink** and **cosk**, are independent of transpose/priming. Similarly for those of j -hat and i -hat, **sinj** and **cosj** for j -hat and **sini** and **cosi** for i -hat because of similarity and congruence. Proof for transpose independence of **cosk** and **sink** is

$$\begin{aligned} \mathbf{cosk}\theta &= \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{so that} & \mathbf{cosk}'\theta = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{cosk}\theta \\ \mathbf{sink}\theta &= \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} & \text{so that} & \mathbf{sink}'\theta = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}^T = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} = \mathbf{sink}\theta \end{aligned}$$

demonstrating that both **cosk** and **sink** are independent of transpose unlike $k \times$ and consequently also unlike $\mathbf{exp}k$. Transpose independence holds true also for **coskj**, **sinj**, **cosi**, and **sini** for reasons of similarity and congruence and so with similar proof. For the $j \times$ basis harmonics

$$\begin{aligned} \mathbf{cosj}\theta &= \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} & \text{so that} & \mathbf{cosj}'\theta = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} = \mathbf{cosj}\theta \\ \mathbf{sinj}\theta &= \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sin\theta \end{pmatrix} & \text{so that} & \mathbf{sinj}'\theta = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sin\theta \end{pmatrix}^T = \begin{pmatrix} \sin\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sin\theta \end{pmatrix} = \mathbf{sinj}\theta \end{aligned}$$

and for the $i \times$ basis harmonics

$$\begin{aligned} \mathbf{cosi}\theta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} & \text{so that} & \mathbf{cosi}'\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} = \mathbf{cosi}\theta \\ \mathbf{sini}\theta &= \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \sin\theta \end{pmatrix} & \text{so that} & \mathbf{sini}'\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \sin\theta \end{pmatrix}^T = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \sin\theta \end{pmatrix} = \mathbf{sini}\theta \end{aligned}$$

It is then intuition to see that the k -hat basis exponential $\mathbf{exp}k$ is not independent of transpose/priming because of the $k \times$ dependence in the Euler form.

Theorem: R^3 Linear algebra, which consists of row vectors, column vectors, and matrices, is complex number mathematics of **unprimed and primed cross products** of the **standard basis**.

Corollary: R^3 Linear algebra is comprised of two cross product triangles, the **unprimed standard cross product triangle** (unprimed complex numbers) and the **primed cross product triangle** (primed complex numbers).

Corollary: These two cross product triangles are sets of complex numbers with the quintessential features of $(\hat{k} \times)^2 = -1$ for unprimed basis vectors ($\hat{i}$ and $\hat{j}$) and $(\times \hat{k}')^2 = -1$ for primed basis vectors ($\hat{i}'$ and $\hat{j}'$); the **unprimed standard cross product triangle** are **unprimed left-to-right ($\rightarrow$) complex numbers [rotation operators]** and the **primed cross product triangle** are **primed right-to-left ($\leftarrow$) complex numbers [rotation operators]**.

Noting that, for the time being, **priming** and **transpose** are **synonymous**. Priming $k \times$ by taking the transpose, as was shown

$$(\hat{k} \times (\cdot))' = (\times (\cdot))' (\hat{k})' = ((\cdot)' \times') \hat{k}' = ((\cdot)' \times') \hat{k}' = (\cdot) \times \hat{k}' \quad \text{where } \times = \times'$$

Note that I was thinking of leaving this $(\times')$ primed as an indicator of operator direction but have now decided against this since it is implied by the primed order of the basis components. It is now understood that $(\cdot)' = (\cdot)$ is a functional argument on primed vectors [and so no primed required on $\times$], that is, operating from the right to a primed vector on the left

$$(\cdot)' = (\cdot, \cdot, \cdot)' = (\hat{i}, \hat{j}, \hat{k})' = (\hat{i}', \hat{j}', \hat{k}')$$

The need to sort these directions became very clear to me whilst working with my TI-89. I am grateful to my calculator for this... yeah, I love my TI-89... which programs similarly to MATLAB... and boots straight away unlike my computers. Now that the directions are sorted (of the primed cross-product triangle), this writing is now in a position to go through the diadic expansion of the primed basis exponential **expk'** in full detail.

The exponential of the primed k -cross is

$$\exp((\cdot) \times \hat{k}' \theta) = (\cdot) \exp(\times \hat{k}' \theta)$$

which is a **right-to-left** ($\leftarrow$) operator for the primed basis components, k' -hat, j' -hat, and i' -hat, similarly as the primed basis-cross, k' -cross $(\hat{k} \times)' = \times \hat{k}'$. Expanding in a power series

$$\exp(\times \hat{k}' \theta) = 1 + (\times \hat{k}') \theta + \frac{1}{2!} (\times \hat{k}')^2 \theta^2 + \frac{1}{3!} (\times \hat{k}')^3 \theta^3 + \dots$$

where each term is an operator from the right for the primed basis components applied at the left. In how I have chosen **priming**, here primed means **row basis** vectors. **Column basis** vectors, on the other hand, are **unprimed** and are hence operated on by **unprimed** operators.

The row basis i' -hat direction

$$\begin{aligned} \hat{i}' \exp(\times \hat{k}' \theta) &= \hat{i}' + \hat{i}' (\times \hat{k}') \theta + \frac{1}{2!} \hat{i}' (\times \hat{k}')^2 \theta^2 + \frac{1}{3!} \hat{i}' (\times \hat{k}')^3 \theta^3 + \dots \\ &= \hat{i}' + \hat{j}' \theta - \frac{1}{2!} \hat{i}' \theta^2 - \frac{1}{3!} \hat{j}' \theta^3 + \dots \\ &= \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots\right) \hat{i}' + \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right) \hat{j}' \\ &= (\cos \theta) \hat{i}' + (\sin \theta) \hat{j}' \end{aligned} \quad \text{where } \hat{i}' (\times \hat{k}') = \hat{j}' \text{ and } \hat{i}' (\times \hat{k}')^2 = \hat{j}' (\times \hat{k}') = -\hat{i}'$$

The row basis j' -hat direction

$$\begin{aligned} \hat{j}' \exp(\times \hat{k}' \theta) &= \hat{j}' + \hat{j}' (\times \hat{k}') \theta + \frac{1}{2!} \hat{j}' (\times \hat{k}')^2 \theta^2 + \frac{1}{3!} \hat{j}' (\times \hat{k}')^3 \theta^3 + \dots \\ &= \hat{j}' - \hat{i}' \theta - \frac{1}{2!} \hat{j}' \theta^2 + \frac{1}{3!} \hat{i}' \theta^3 + \dots \\ &= \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots\right) \hat{i}' + \left(1 - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots\right) \hat{j}' \\ &= (-\sin \theta) \hat{i}' + (\cos \theta) \hat{j}' \end{aligned} \quad \text{where } \hat{j}' \times \hat{k}' = -\hat{i}' \text{ and } \hat{j}' (\times \hat{k}')^2 = -\hat{i}' (\times \hat{k}') = -\hat{j}'$$

The row basis k' -hat direction

$$\begin{aligned} \hat{k}' \exp(\times \hat{k}' \theta) &= \hat{k}' + \hat{k}' \times \hat{k}' \theta + \frac{1}{2!} \hat{k}' (\times \hat{k}')^2 \theta^2 + \frac{1}{3!} \hat{k}' (\times \hat{k}')^3 \theta^3 + \dots \\ &= \hat{k}' + 0 \hat{k}' + 0 \hat{k}' + 0 \hat{k}' + \dots \\ &= \hat{k}' \end{aligned} \quad \text{where } \hat{k}' \times \hat{k}' = 0 \hat{k}' \quad \hat{k}' (\times \hat{k}')^n = 0 \hat{k}'$$

In previous documents, specifically the first k -cross and k -hat exponential, when **diadically expanding** from the vector components for the operator in the matrix basis, this was for a **column basis expansion**. This operates as a matrix basis onto a column basis from the left (the matrix operator) to the right (the column operand), and so the component of each **direction** in the column (unprimed) components **maps to columns** of the diadically expanded matrix basis. Here the **diadic expansion** is for a **row basis**, the components of each direction in the row (primed) components **map to rows**. Now with the confidence that it works, I can now state it as a theorem.

Theorem: (1) diadic expansion of unprimed (column vector) direction components map to columns in the matrix basis and (2) diadic expansions of primed (row vector) direction components map to rows in the matrix basis.

Corollary: R^3 linear algebra column vector multiplication ($3 \text{by} 1 \times 3 \text{by} 1$) is a cross product, that is, sum of complex multiplications. R^3 linear algebra row vector multiplication ($1 \text{by} 3 \times 1 \text{by} 3$) is a cross product with the cross product triangle reversed, that is, sum of

complex multiplications [which can be seen by recognizing the left to right complex/rotation operators as primed/conjugate complex numbers]. This similarly defines (3by1 $\times$ 3by3) and (3by3 $\times$ 1by3) complex-quaternion-like cross products.

This is for similar reasons as these operators row or column reducing a matrix in pivotal condensation.

So the components taken together in a position (or colour) row vector for

$$((\cdot)\hat{i}', (\cdot)\hat{j}', (\cdot)\hat{k}') \exp(\times \hat{k}' \theta)$$

then by diadic expansion result in

$$\begin{aligned} ((\cdot)\hat{i}', (\cdot)\hat{j}', (\cdot)\hat{k}') \exp(\times \hat{k}' \theta) &= \hat{i}' \hat{i} ((\cos \theta) \hat{i}' + (\sin \theta) \hat{j}') + \hat{j}' \hat{j} ((-\sin \theta) \hat{i}' + (\cos \theta) \hat{j}') + \hat{k}' \hat{k} (\hat{k}') \\ &= (\cdot) \hat{i}' ((\cos \theta) \hat{i}' \hat{i}' + (\sin \theta) \hat{i}' \hat{j}') + (\cdot) \hat{j}' ((-\sin \theta) \hat{j}' \hat{i}' + (\cos \theta) \hat{j}' \hat{j}') + (\cdot) \hat{k}' (1 \hat{k}' \hat{k}') \\ &= ((\cdot)\hat{i}', (\cdot)\hat{j}', (\cdot)\hat{k}') \begin{pmatrix} (\cos \theta) \hat{i}' \hat{i}' & (\sin \theta) \hat{i}' \hat{j}' & 0 \hat{i}' \hat{k}' \\ (-\sin \theta) \hat{j}' \hat{i}' & (\cos \theta) \hat{j}' \hat{j}' & 0 \hat{j}' \hat{k}' \\ 0 \hat{k}' \hat{i}' & 0 \hat{k}' \hat{j}' & 1 \hat{k}' \hat{k}' \end{pmatrix} \end{aligned}$$

that is,

$$\exp(\times \hat{k}' \theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{nb., actually } ((\cdot) \quad (\cdot) \quad (\cdot)) \exp(\times \hat{k}' \theta) = ((\cdot) \quad (\cdot) \quad (\cdot)) \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the conjugate k -hat exponential **expk'** as per the earlier bit of mathematics. As a rotation matrix, this is the same as the inverse and synonymously transpose since it is the **anti-sense rotation** of the k -hat exponential **expk**, but is a **right-to-left operator**. This also demonstrates that the **inverse** of a basis-hat exponential is **technically distinct** from its **transpose** (primed) counterpart. Transpose [priming] changes the direction of the operator, that is, whether it operates on a **column basis position vector** or a **row basis position vector (or colour vector)**. Inverse does not, that is, does not change the direction of the operator. Although in most standard linear algebra this is just a triviality, transpose and inverse are the defining features for the rotation matrix in R^3 so it may be relevant at some point in the future and some specialized situations including for keeping note of the direction for which the rotation matrix applies as an operator.

The Euler form of the primed basis-hat exponential **expk'** is

$$\begin{aligned} \exp(\times \hat{k}' \theta) &= (\cdot) \exp(\times \hat{k}' \theta) = (\cdot) \mathbf{cisk}' \theta \\ &= (\cdot) \mathbf{cosk}' \theta + (\cdot) \times \hat{k}' \mathbf{sink}' \theta \\ &= (\cdot) \mathbf{cosk} \theta + (\cdot) \times \hat{k}' \mathbf{sink} \theta \\ &= \mathbf{cosk} \theta + (\times \hat{k}' \mathbf{sink} \theta) \end{aligned}$$

where $\mathbf{cosk}' \theta = \mathbf{cosk} \theta$ and $\mathbf{sink}' \theta = \mathbf{sink} \theta$. Since we read from **left-to-right**, in word and in maths, it may be more convenient to leave the conjugate of $k \times$ as a primed operator, so

$$\exp(\times \hat{k}' \theta) = \mathbf{cosk} \theta + (\hat{k} \times)' \mathbf{sink} \theta \quad \text{rather than} \quad \exp(\times \hat{k}' \theta) = \mathbf{cosk} \theta + (\times \hat{k}') \mathbf{sink} \theta$$

where the priming on $k \times$ implies the **right-to-left** direction. In scalar complex algebra, this corresponds to the sign change of the conjugate in a complex conjugate pair. This could also be written, for convenience in some situations, as

$$\exp(\theta (\hat{k} \times)') = \mathbf{cosk} \theta + (\mathbf{sink} \theta \hat{k} \times)'$$

and given the transpose independence of **sink**, it could even be written for convenience as

$$\exp(\times \hat{k}' \theta) = \mathbf{cosk} \theta + (\mathbf{sink} \theta) (\hat{k} \times)'$$

Instead of the diadic expansion for the row bases, the rotation matrix form can also be obtained as the first section of mathematics at the start of this document by multiplying by **expk'** (rather than **expk**) from the right (rather than the left)

$$\exp(\times \hat{k}' \theta) = I e^{\times \hat{k}' \theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{\times \hat{k}' \theta} = \begin{pmatrix} e^{\times \hat{k}' \theta} & 0 & 0 \\ 0 & e^{\times \hat{k}' \theta} & 0 \\ 0 & 0 & e^{\times \hat{k}' \theta} \end{pmatrix} = \begin{pmatrix} 1 \hat{i} \hat{i}' & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & 1 \hat{j} \hat{j}' & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & 1 \hat{k} \hat{k}' \end{pmatrix} e^{\times \hat{k}' \theta} = \begin{pmatrix} \hat{i} \hat{i}' e^{\times \hat{k}' \theta} & 0 \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ 0 \hat{j} \hat{i}' & \hat{j} \hat{j}' e^{\times \hat{k}' \theta} & 0 \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & \hat{k} \hat{k}' e^{\times \hat{k}' \theta} \end{pmatrix}$$

and then expanding out each of the three conjugate basis exponentials into Euler form **expk'** $\theta = \mathbf{cisk}' \theta$ with diad decomposition into **cosk** θ and **sink** θ

$$\begin{aligned} \exp(\times \hat{k}' \theta) &= \mathbf{cosk} \theta + (\times \hat{k}' \mathbf{sink} \theta) \\ &= \mathbf{cosk} \theta + (\hat{k} \times)' \mathbf{sink} \theta \end{aligned}$$

doing the **right-to-left** component (complex number) cross products on the i' -hat, j' -hat, and k' -hat components and then rearranging in the matrix lattice. Bit disconcerting, mind you, working **right-to-left**.

This technique amounts to the same thing as a diadic expansion in the row basis except it is more general for circumstances such as for arbitrary **Hermitian matrices** where a conjugate basis exponential is operating on one or more of the matrix basis components other than those exclusively and homogeneously (in $k \times$) on the diagonal.

When the matrix basis components are visible and the **scalar harmonics** and **rotation eigenvalue** are matched/coupled with their row basis components (that is, **bracketed scale factors** with their associated basis components), the conjugate k -cross basis exponential **expk'** is

$$\exp(\times \hat{k}' \theta) = \begin{pmatrix} \hat{i}(\hat{i}' \cos \theta) & \hat{i}(\hat{j}' \sin \theta) & \hat{i}(\hat{k}' 0) \\ \hat{j}(-\hat{i}' \sin \theta) & \hat{j}(\hat{j}' \cos \theta) & \hat{j}(\hat{k}' 0) \\ \hat{k}(\hat{i}' 0) & \hat{k}(\hat{j}' 0) & \hat{k}(\hat{k}' 1) \end{pmatrix}$$

noting for above that the **scalar sines and cosines** are coupled with the **primed basis components** (rather than **unprimed basis components** as earlier) for the primed (conjugate) k -hat basis exponential **expk'**. Keeping the scale factors bracketed and coupled with the relevant vector base (either column or row base) within the matrix basis then indicates whether the basis exponential in rotation matrix format has been diadically expanded as an operator for either (1) a **row position (or colour) vector** or (2) a **column position (or colour) vector** ... that is, expanded for either (1) a **row position (or colour) vector** as a **right-to-left (primed) operator** or (2) a **column position (or colour) vector** as a **left-to-right (unprimed) operator**. This also then explicitly shows the difference between the transpose and inverse when otherwise it seems as though it is just a sign change on the scalar sines. The inverse of the conjugate (primed) basis exponential **expk'** is

$$\exp(\times \hat{k}' \theta)^{-1} = \begin{pmatrix} \hat{i}(\hat{i}' \cos \theta) & \hat{i}(-\hat{j}' \sin \theta) & \hat{i}(\hat{k}' 0) \\ \hat{j}(\hat{i}' \sin \theta) & \hat{j}(\hat{j}' \cos \theta) & \hat{j}(\hat{k}' 0) \\ \hat{k}(\hat{i}' 0) & \hat{k}(\hat{j}' 0) & \hat{k}(\hat{k}' 1) \end{pmatrix}$$

whereas for the unprimed basis exponential **expk**

$$\exp(\theta \hat{k} \times) = \begin{pmatrix} (\hat{i} \cos \theta) \hat{i}' & (-\hat{i} \sin \theta) \hat{j}' & (\hat{i} 0) \hat{k}' \\ (\hat{j} \sin \theta) \hat{i}' & (\hat{j} \cos \theta) \hat{j}' & (\hat{j} 0) \hat{k}' \\ (\hat{k} 0) \hat{i}' & (\hat{k} 0) \hat{j}' & (\hat{k} 1) \hat{k}' \end{pmatrix} \quad \text{it's inverse is} \quad \exp(\theta \hat{k} \times)^{-1} = \begin{pmatrix} (\hat{i} \cos \theta) \hat{i}' & (\hat{i} \sin \theta) \hat{j}' & (\hat{i} 0) \hat{k}' \\ (-\hat{j} \sin \theta) \hat{i}' & (\hat{j} \cos \theta) \hat{j}' & (\hat{j} 0) \hat{k}' \\ (\hat{k} 0) \hat{i}' & (\hat{k} 0) \hat{j}' & (\hat{k} 1) \hat{k}' \end{pmatrix}$$

Transpose, on the other hand, reverses the ordering of the bracketing when the brackets are kept coupled with the basis component for which it was diadically expanded. $\exp(\theta \hat{k} \times) = \exp(\theta (\hat{k} \times)') = \exp(\times \hat{k}' \theta)$

$$\exp(\theta \hat{k} \times)' = \begin{pmatrix} (\hat{i} \cos \theta) \hat{i}' & (-\hat{i} \sin \theta) \hat{j}' & (\hat{i} 0) \hat{k}' \\ (\hat{j} \sin \theta) \hat{i}' & (\hat{j} \cos \theta) \hat{j}' & (\hat{j} 0) \hat{k}' \\ (\hat{k} 0) \hat{i}' & (\hat{k} 0) \hat{j}' & (\hat{k} 1) \hat{k}' \end{pmatrix}' = \begin{pmatrix} \hat{i}(\hat{i}' \cos \theta) & \hat{i}(\hat{j}' \sin \theta) & \hat{i}(\hat{k}' 0) \\ \hat{j}(-\hat{i}' \sin \theta) & \hat{j}(\hat{j}' \cos \theta) & \hat{j}(\hat{k}' 0) \\ \hat{k}(\hat{i}' 0) & \hat{k}(\hat{j}' 0) & \hat{k}(\hat{k}' 1) \end{pmatrix}$$

In most instances, the component bracketing is just a triviality (as for the sign change) although there may be times where it proves essential to maintain this bracketing notation as an indicator of whether the operator is a **left-to-right operator** for a **column position vector** or a **right-to-left operator** for a **row position vector**. In a future document I will be writing, the **bracketing** is essential to **maintain colour orientation**. Without the bracketing, only positions can be tracked. Loss of bracketing will result in loss of colour orientation.

This is also another clear visual reminder that the **components of the matrix basis** occur from the **outer product** which expands the **vector bases** into the **matrix basis**. The outer product showing by choice of the **order positions** of the **bracketed scalars**, that is, the bracketed positions of the scalar sines, scalar cosines, zeroes, and the unity rotation eigenvalue. This is relevant because of the fact that one cannot just **outer product expand** vectors into **expk** (the rotation matrix) and this requires **diadic expansion** and the **bracketing** then shows whether the diadic expansion was for a **column basis** or a **row basis** (and hence whether it was expanded to operate on a row or column position (or colour) vector).

It is also relevant to changes of base between various coordinate systems.

This also shows clearly the components of matrix-matrix products are then each inner products when operating on the position vector with the component column and row vector bases bracketed with the scale factors, the inner products showing by choice of the inner positions of the scalars when multiplication takes place which keeps the ordering of the bracketing positions. There, of course, can be long chains of these. Trivial becomes very non-trivial.

As another point of interest, in terms of c-programming, the **basis components** are essentially the **memory addresses**. Bracketing keeps the scalars associated with their basis components which is essentially keeping the knowledge explicit of a data item with its memory address when vector expanded into a matrix basis. These can be accessed and manipulated with linked lists. At some stage, this may make for more sophistication in processors... providing, for example, a means to give the billions of transistors more dynamic. Given that the number of transistors on a chip is reaching its quantum limit, I envisage that in the future in order to advance the technology, regions of processor cores could be engineered into Rubiks quaternions.

Also, whilst on the topic of computer programming, the cross product symbol/operator $\times$ is not currently featured in ASCII (American Standard Code for Information Interchange). Given that all 128 positions are currently full and expanding it to 256 would be inefficient, I think that it would be useful for mathematical programmers for $\times$ to be substituted into ASCII with one of the lesser items of importance (if a redundancy in the 128 items can be found and agreed upon). It is interesting that $(*)$ are the nodes of a hexagon, but I specifically require $(\times)$. I required this $(\times)$ but it is currently not available.

3. Row Reduction and Column Reduction for Pivotal Condensation

This section discusses **row** and **column** reductions (pivotal condensation) by the technique of applying unprimed and primed basis cross products. This is the workings of **Gauss pivotal condensation** to **echelon forms** by use of complex numbers [rotation operators], that is, with $i = \hat{k} \times$ (and $\hat{j} \times$ and $\hat{i} \times$) for **row reductions** and $i' = (\hat{k} \times)'$ (and $(\hat{j} \times)'$ and $(\hat{i} \times)'$) for **column reductions** respectively.

As was noted in k -cross and k -hat exponential, the operators $k \times$, $j \times$, and $i \times$ as unprimed operators can be used to **row reduce** an R^3 matrix to echelon and row reduced echelon form. These operators do so as **left-to-right operators** on the matrix basis. Equivalently, the primed operators can be used to **column reduce** and do so as **right-to-left operators** on the matrix basis. This small piece of mathematics shows why **row reduction** and **column reduction** can be carried out similarly. It is therefore a mathematical proof in terms of the basis functions, which although somewhat trivial, shows how it works. It is also the workings of obtaining the minimum basis cross product factorization of an R^3 matrix (explaining and extending the Cayley-Hamilton theorem for R^3 ... that is, when I start doing the calculus).

As for the previous section, priming has required the recognition that when working and writing out the mathematics of primed operators [basis variables appended with $\times$], the direction of the **cross product triangle** is reversed. This is reading their **cyclic properties** in the other direction similarly as being **right-to-left operators** rather than **left-to-right operators** of the unprimed complex numbers [unprimed basis cross-products].

Either **row reduction** or **column reduction** can be shown by

- (1) applying to the **unprimed vector basis** in the matrix being row reduced, from **left-to-right**, the row operation with $k \times$ or
- (2) as a priming of a **primed matrix** (transposing it) being operated on from the **right-to-left** by $(k \times)' = \times k'$.

The working matrix is

$$A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix} \hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times = \hat{k} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Applying $k \times$ from the left to A , $\hat{k} \times A$, to **row reduce** the matrix with linear combination then crosses out the k -hat-row, changes the sign of the j -hat row, and then permutes it with the i -hat-row with the **crossing-sign-change** of the j -hat row (which, of relevance, gives the operator the property of quartic roots of unity):

$$(1) \hat{k} \times A = \hat{k} \times \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix} = \begin{pmatrix} -\mathbf{d} & -\mathbf{e} & -\mathbf{f} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ 0 & 0 & 0 \end{pmatrix}$$

Priming the row reduction operation (with transpose), that is priming both the operator and the operand, then $(\hat{k} \times A)' = A'(\hat{k} \times)' = A'(\times' \hat{k}') = A'(\times \hat{k}')$ which is then primed k -cross (a column operator) operating on (from the left) transposed A which then carries out the equivalent column operation on the transpose of A , that is, crosses out the k' -hat-column, changes the sign of the j' -hat-column, and then permutes it with the i' -hat-column. This has hence converted the operator from that used for the **row reduction** to that used for a **column reduction** demonstrating equivalence:

$$(2) (\hat{k} \times A)' = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix}' \times \hat{k}' = \begin{pmatrix} \mathbf{a} & \mathbf{d} & \mathbf{g} \\ \mathbf{b} & \mathbf{e} & \mathbf{h} \\ \mathbf{c} & \mathbf{f} & \mathbf{i} \end{pmatrix} \times \hat{k}' = \begin{pmatrix} \mathbf{a} & \mathbf{d} & \mathbf{g} \\ \mathbf{b} & \mathbf{e} & \mathbf{h} \\ \mathbf{c} & \mathbf{f} & \mathbf{i} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\mathbf{d} & \mathbf{a} & 0 \\ -\mathbf{e} & \mathbf{b} & 0 \\ -\mathbf{f} & \mathbf{c} & 0 \end{pmatrix}$$

The cross product maths for left to right (k -cross, $k \times$) **row reduction** is

$$(1) \hat{k} \times A = \begin{pmatrix} \hat{k} \times \\ \mathbf{a} \hat{i} \hat{i}' & \mathbf{b} \hat{i} \hat{j}' & \mathbf{c} \hat{i} \hat{k}' \\ \mathbf{d} \hat{j} \hat{i}' & \mathbf{e} \hat{j} \hat{j}' & \mathbf{f} \hat{j} \hat{k}' \\ \mathbf{g} \hat{k} \hat{i}' & \mathbf{h} \hat{k} \hat{j}' & \mathbf{i} \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} \mathbf{a} \hat{k} \times \hat{i} \hat{i}' & \mathbf{b} \hat{k} \times \hat{i} \hat{j}' & \mathbf{c} \hat{k} \times \hat{i} \hat{k}' \\ \mathbf{d} \hat{k} \times \hat{j} \hat{i}' & \mathbf{e} \hat{k} \times \hat{j} \hat{j}' & \mathbf{f} \hat{k} \times \hat{j} \hat{k}' \\ \mathbf{g} \hat{k} \times \hat{k} \hat{i}' & \mathbf{h} \hat{k} \times \hat{k} \hat{j}' & \mathbf{i} \hat{k} \times \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} -\mathbf{d} \hat{i} \hat{i}' & -\mathbf{e} \hat{i} \hat{j}' & -\mathbf{f} \hat{i} \hat{k}' \\ \mathbf{a} \hat{j} \hat{i}' & \mathbf{b} \hat{j} \hat{j}' & \mathbf{c} \hat{j} \hat{k}' \\ 0 \hat{k} \hat{i}' & 0 \hat{k} \hat{j}' & 0 \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} -\mathbf{d} & -\mathbf{e} & -\mathbf{f} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ 0 & 0 & 0 \end{pmatrix}$$

where $(\hat{k} \times) \hat{i} = \hat{j}$, $(\hat{k} \times) \hat{j} = -\hat{i}$ and $(\hat{k} \times) \hat{k} = 0 \hat{k}$.

Theorem: The operation $(\hat{k} \times)A$ defines a $3\text{by}1 \times 3\text{by}3$ vector-matrix product. Such an operation may be called left-to-right quaternion product (complex number product when the cross operator has a single component as here for $k \times$). The $3\text{by}1$ appended with $\times$ is diadically expanded into a $3\text{by}3$ so that the left-to-right quaternion product is implemented with a $3\text{by}3 * 3\text{by}3$ with an ordinary matrix multiplication.

For $(\hat{k} \times A)' = A'(\hat{k} \times)' = A'(\times' \hat{k}') = A'(\times \hat{k}')$ the cross product maths for right to left (primed k -cross, $(k \times)'$) **column reduction** is

$$(2) (\hat{k} \times A)' = \begin{pmatrix} \mathbf{a} \hat{i} \hat{i}' & \mathbf{d} \hat{i} \hat{j}' & \mathbf{g} \hat{i} \hat{k}' \\ \mathbf{b} \hat{j} \hat{i}' & \mathbf{e} \hat{j} \hat{j}' & \mathbf{h} \hat{j} \hat{k}' \\ \mathbf{c} \hat{k} \hat{i}' & \mathbf{f} \hat{k} \hat{j}' & \mathbf{i} \hat{k} \hat{k}' \end{pmatrix} (\times \hat{k}') = \begin{pmatrix} \mathbf{a} \hat{i} \hat{i}' \times \hat{k}' & \mathbf{d} \hat{i} \hat{j}' \times \hat{k}' & \mathbf{g} \hat{i} \hat{k}' \times \hat{k}' \\ \mathbf{b} \hat{j} \hat{i}' \times \hat{k}' & \mathbf{e} \hat{j} \hat{j}' \times \hat{k}' & \mathbf{h} \hat{j} \hat{k}' \times \hat{k}' \\ \mathbf{c} \hat{k} \hat{i}' \times \hat{k}' & \mathbf{f} \hat{k} \hat{j}' \times \hat{k}' & \mathbf{i} \hat{k} \hat{k}' \times \hat{k}' \end{pmatrix} = \begin{pmatrix} -\mathbf{d} \hat{i} \hat{i}' & \mathbf{a} \hat{i} \hat{j}' & 0 \hat{i} \hat{k}' \\ -\mathbf{e} \hat{j} \hat{i}' & \mathbf{b} \hat{j} \hat{j}' & 0 \hat{j} \hat{k}' \\ -\mathbf{f} \hat{k} \hat{i}' & \mathbf{c} \hat{k} \hat{j}' & 0 \hat{k} \hat{k}' \end{pmatrix} = \begin{pmatrix} -\mathbf{d} & \mathbf{a} & 0 \\ -\mathbf{e} & \mathbf{b} & 0 \\ -\mathbf{f} & \mathbf{c} & 0 \end{pmatrix}$$

where $\hat{i}'(\times \hat{k}') = \hat{j}'$, $\hat{j}'(\times \hat{k}') = -\hat{i}'$ and $\hat{k}'(\times \hat{k}') = 0 \hat{k}'$ where the primed operators **reverse the order** of the **cross-product triangle**, that is, since they **operate right-to-left** when **viewed from left to right** appear so as the **direction of the cross product triangle** has been **reversed**.

Theorem: The operation $A(\hat{k} \times)' = A(\times \hat{k}')$ defines a $3\text{by}3 \times 1\text{by}3$ matrix-vector product. Such an operation may be called right-to-left (conjugate) quaternion product (complex number product when the cross operator has a single component as above for $\times \hat{k}'$).

Note that the reduction by use of linear combinations of $k \times$, $j \times$, and $i \times$ exploits the fact that $k \times$, $j \times$, and $i \times$ are singular.

Since the linear combination of **unprimed basis crosses** ($k \times$, $j \times$, $i \times$) that **row reduce** a matrix to the identity, the operators that bring the matrix to reduced echelon form then forms the **basis-cross-product-factorisation** for the inverse of that matrix. This can then be inverted for the basis factorisation of the matrix. This piece of mathematics shows that the equivalent **basis-cross-product-factorisation** can be expressed in terms of the **primed basis crosses** ($\times \hat{k}'$, $\times \hat{j}'$, $\times \hat{i}'$) and demonstrates how one would do this, notably, find a combination of the primed-basis-rotation/curls that **column reduce** the matrix to the identity. This combination of the basis-curls is then the **basis-cross-product-factorisation** for the **inverse** and for which can be inverted for the **cross-product basis-factorisation** of the **matrix** itself (the inverse of the inverse). When this polynomial in $k \times$ has minimal factors, it then forms the minimum polynomial. I'll be doing all the maths with eigenvalues in a latter planned lot of work [although, as here, it doesn't actually need eigenvalues... I have been doing all of this work as preparation to eigenvalues and calculus... this is purely algebraic, eigenvalues are calculus].

Equivalently, other than the transpose of the matrix being reduced, priming the operation of A operated on by primed k -cross $(A \times \hat{k}')' = (\times \hat{k}')' A' = (\hat{k}')' (\times)' A' = \hat{k}' \times A'$ which is then unprimed k -cross operating (from the right) on transposed A which has hence converted operator from that used for the **column reduction** to that used for a **row reduction**. These are

$$\hat{k} \times A' = \hat{k} \times \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix}' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{d} & \mathbf{g} \\ \mathbf{b} & \mathbf{e} & \mathbf{h} \\ \mathbf{c} & \mathbf{f} & \mathbf{i} \end{pmatrix} = \begin{pmatrix} -\mathbf{b} & -\mathbf{e} & -\mathbf{h} \\ \mathbf{a} & \mathbf{d} & \mathbf{g} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A(\hat{k} \times) = A \times \hat{k}' = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix} \times \hat{k}' = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{i} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\mathbf{b} & \mathbf{a} & 0 \\ -\mathbf{e} & \mathbf{d} & 0 \\ -\mathbf{h} & \mathbf{g} & 0 \end{pmatrix}$$

The mathematics demonstrating $A \times \hat{k}'$ and $(A \times \hat{k}')' = (\times \hat{k}')' A' = (\hat{k}'' \times') A' = (\hat{k} \times) A'$ with the cross products on the basis components in the matrix basis is the same for the rearranged scalar components of A and so is omitted.

This small bit of mathematics has shown why **row reduction** and **column reduction** can be carried out similarly with linear combinations of **unprimed (left-to-right $\rightarrow$)** and **primed (right-to-left $\leftarrow$)** basis-cross-products respectively. It demonstrates further that it is the single component basis cross products [complex numbers/quaternions] that do this (and implemented in diadically expanded form). This potentially provides another relevant item of theory for [teaching, learning, and doing] linear algebra. For my purposes, it provides the required basis in which to do calculus within volumes in R^3 and shows how it all works for an extension to Rubiks quaternions. It also demonstrates that linear algebra (for the time being, in R^3 and the subset R^2) is a system of complex numbers and so consequently, the separate fields of R^3 linear algebra and complex algebra are now unified.

4. Diadic Mathematics – Matrix Transpose

Here I outline the diadic mathematics which demonstrates the mechanics of transpose, similarly to that of matrix multiplication. If for nothing else, it is interesting. The working matrix is

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a(\hat{i}\hat{i}') & b(\hat{i}\hat{j}') & c(\hat{i}\hat{k}') \\ d(\hat{j}\hat{i}') & e(\hat{j}\hat{j}') & f(\hat{j}\hat{k}') \\ g(\hat{k}\hat{i}') & h(\hat{k}\hat{j}') & i(\hat{k}\hat{k}') \end{pmatrix}$$

and in **full basis expanded form** this matrix is

$$A = a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}'$$

Taking the transpose of both sides

$$\begin{aligned} A' &= (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}' + d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}' + g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}')' && \text{transpose of sum equal to sum of transposes} \\ &= a(\hat{i}\hat{i}')' + b(\hat{i}\hat{j}')' + c(\hat{i}\hat{k}')' + d(\hat{j}\hat{i}')' + e(\hat{j}\hat{j}')' + f(\hat{j}\hat{k}')' + g(\hat{k}\hat{i}')' + h(\hat{k}\hat{j}')' + i(\hat{k}\hat{k}')' \\ &= a(\hat{i}')'\hat{i}' + b(\hat{j}')'\hat{i}' + c(\hat{k}')'\hat{i}' + d(\hat{i}')'\hat{j}' + e(\hat{j}')'\hat{j}' + f(\hat{k}')'\hat{j}' + g(\hat{i}')'\hat{k}' + h(\hat{j}')'\hat{k}' + i(\hat{k}')'\hat{k}' \\ &= a\hat{i}\hat{i}' + b\hat{j}\hat{i}' + c\hat{k}\hat{i}' + d\hat{i}\hat{j}' + e\hat{j}\hat{j}' + f\hat{k}\hat{j}' + g\hat{i}\hat{k}' + h\hat{j}\hat{k}' + i\hat{k}\hat{k}' \\ &= \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} a(\hat{i}\hat{i}') & d(\hat{i}\hat{j}') & g(\hat{i}\hat{k}') \\ b(\hat{j}\hat{i}') & e(\hat{j}\hat{j}') & h(\hat{j}\hat{k}') \\ c(\hat{k}\hat{i}') & f(\hat{k}\hat{j}') & i(\hat{k}\hat{k}') \end{pmatrix} \end{aligned}$$

which is the transpose of A .

Transpose of the **row basis expanded form** returns the transpose of A in a **column basis expanded form** and visa versa. The transpose of the **column basis expanded form** returns the transpose of A in the **row basis expanded form**. This may prove useful in certain circumstances. Such instances include row and column reductions.

The matrix basis factorised into **row basis expanded form** is

$$\begin{aligned} A &= (a\hat{i}\hat{i}' + b\hat{i}\hat{j}' + c\hat{i}\hat{k}') + (d\hat{j}\hat{i}' + e\hat{j}\hat{j}' + f\hat{j}\hat{k}') + (g\hat{k}\hat{i}' + h\hat{k}\hat{j}' + i\hat{k}\hat{k}') \\ &= \hat{i}(a\hat{i}' + b\hat{j}' + c\hat{k}') + \hat{j}(d\hat{i}' + e\hat{j}' + f\hat{k}') + \hat{k}(g\hat{i}' + h\hat{j}' + i\hat{k}') \end{aligned}$$

and its transpose is

$$A' = \left(\hat{i} (a \hat{i}' + b \hat{j}' + c \hat{k}') + \hat{j} (d \hat{i}' + e \hat{j}' + f \hat{k}') + \hat{k} (g \hat{i}' + h \hat{j}' + i \hat{k}') \right)'$$

$$= (a \hat{i} + b \hat{j} + c \hat{k}) \hat{i}' + (d \hat{i} + e \hat{j} + f \hat{k}) \hat{j}' + (g \hat{i} + h \hat{j} + i \hat{k}) \hat{k}'$$

which is the sum of **row vector bases** times **column bases**, that is, the transpose is expressed in **column basis expanded form**. Also, applying $k \times$ from the left to A on the right (which notably results in a 3by3 matrix basis after a 3by1 $\times$ 3by3 quaternion product)

$$\hat{k} \times A = \hat{k} \times \hat{i} (a \hat{i}' + b \hat{j}' + c \hat{k}') + \hat{k} \times \hat{j} (d \hat{i}' + e \hat{j}' + f \hat{k}') + \hat{k} \times \hat{k} (g \hat{i}' + h \hat{j}' + i \hat{k}')$$

$$= -\hat{i} (d \hat{i}' + e \hat{j}' + f \hat{k}') + \hat{j} (a \hat{i}' + b \hat{j}' + c \hat{k}') + \hat{k} (0 \hat{i}' + 0 \hat{j}' + 0 \hat{k}')$$

which demonstrates $k \times$ (and $j \times$ and $i \times$) as the working tool for **row reduction** and so demonstrates the working principal of **row reduction** by complex number cross products [rotation operators] in an algebraically pleasing and concise way.

Factorization of A in **column basis expanded form**

$$A = (a \hat{i} \hat{i}' + d \hat{j} \hat{i}' + g \hat{k} \hat{i}') + (b \hat{i} \hat{j}' + e \hat{j} \hat{j}' + h \hat{k} \hat{j}') + (c \hat{i} \hat{k}' + f \hat{j} \hat{k}' + i \hat{k} \hat{k}')$$

$$= (a \hat{i} + d \hat{j} + g \hat{k}) \hat{i}' + (b \hat{i} + e \hat{j} + h \hat{k}) \hat{j}' + (c \hat{i} + f \hat{j} + i \hat{k}) \hat{k}'$$

and its transpose is

$$A' = \left((a \hat{i} + d \hat{j} + g \hat{k}) \hat{i}' + (b \hat{i} + e \hat{j} + h \hat{k}) \hat{j}' + (c \hat{i} + f \hat{j} + i \hat{k}) \hat{k}' \right)'$$

$$= \hat{i} (a \hat{i}' + d \hat{j}' + g \hat{k}') + \hat{j} (b \hat{i}' + e \hat{j}' + h \hat{k}') + \hat{k} (c \hat{i}' + f \hat{j}' + i \hat{k}')$$

which is the sum of **row bases** times **column bases**, that is, the transpose is expressed in **row basis expanded form**. Also, applying $\times k'$ from the right to A on the left (note that this is 3by3 $\times$ 1by3 = 3by3)

$$A \times \hat{k}' = (a \hat{i} + d \hat{j} + g \hat{k}) \hat{i}' \times \hat{k}' + (b \hat{i} + e \hat{j} + h \hat{k}) \hat{j}' \times \hat{k}' + (c \hat{i} + f \hat{j} + i \hat{k}) \hat{k}' \times \hat{k}'$$

$$= -(b \hat{i} + e \hat{j} + h \hat{k}) \hat{i}' + (a \hat{i} + d \hat{j} + g \hat{k}) \hat{j}' + (0 \hat{i} + 0 \hat{j} + 0 \hat{k}) \hat{k}'$$

which demonstrates $(k \times)'$ (and $(j \times)'$ and $(i \times)'$) as the working tool for **column reduction**.

Hence this shows that these forms can also be used for convenience to demonstrate **row and column reductions** and is a main reason that they have seemed worthy of discussing despite being elementary. I also like how it looks as it shows that a matrix is a composition of complex numbers and quaternions.

Theorem (from the first document): a standard R^3 position vector appended with $\times$ is a quaternion and the components are complex numbers (orthogonal axis rotation operators: $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$).

Corollary: A matrix is a composition of these complex numbers (orthogonal axis rotation operators: $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$) applied to a primed vector basis ($\hat{i}'$, $\hat{j}'$, and $\hat{k}'$).

This algebra of transposes and matrix multiplication, as well as **basis cross product factorisations**, demonstrates again that matrix algebra is the algebra of vectors [and hence complex numbers], most particularly, the inner and outer products of the basis vectors [of the complex numbers]. All matrix mathematics is the mathematics of the inner and outer products of vectors (the basis vectors) with the cross operator for **basis crosses** (of the basis vectors) of these vectors as **the working tool of factorization**.

I am of the opinion that all linear algebra can essentially be taught as systems of vectors and diadic expansions since all the mechanics can be proved, and hence taught systematically, in terms of vectors and diads which gives matrix algebra a systematically complete theoretical basis. This is a complementary addition to the standard teaching and learning the systems by wrought and complements how I learned linear algebra by wrought, in terms of the vector axioms, and knowing (absolutely from past experience) that it works. I also think that this will make group theory a fully applied field of mathematics, especially in the context of Rubiks quaternions, given the group properties of $k \times$, $j \times$, and $i \times$.

I fully endorse the use of Rubiks quaternions (Rubiks cubes) in academic and applied mathematics.

I am also stating again that I think that Erno Rubik should be nominated for a Nobel Prize in Physics.

5. Nilpotent Mixed Double Crosses and their Exponentials: $j \times k \times$, $k \times i \times$, and $j \times i \times$ and $\exp jk$, $\exp ki$, and $\exp ji$

Since there are three R^3 axes and so three basis vectors generating these axes, there are three complex numbers once those basis vectors are appended with $\times$, one for each lattice plane of R^3 that intersect orthogonally at the origin (three mutually orthogonal Argand diagrams of real R^3). These complex numbers are single component cross products.

So there are **three complex numbers** ($k \times$, $j \times$, $i \times$) for any vector basis as operators for that space. Similarly, there are **nine second order complex numbers**. These are the **three double crosses** ($k \times k \times$, $j \times j \times$, $i \times i \times$ corresponding to $i^2 = -1$, $j^2 = -1$, and $k^2 = -1$) and **six mixed double crosses** ($i \times j \times$, $i \times k \times$, $j \times i \times$, $j \times k \times$, $k \times i \times$, and $k \times j \times$) as per the tables of second order basis curls, single component cross products and repeated single component cross products that [now] define complex numbers.

The six mixed double crosses ($i \times j \times$, $i \times k \times$, $j \times i \times$, $j \times k \times$, $k \times i \times$, and $k \times j \times$) are nilpotent and these nilpotent operators and the exponentials of these nilpotent operators are the topic of this section of mathematics. The six mixed double cross exponentials are **expij**, **expik**, **expji**, **expjk**, **expki**, and **expkj**, which can be compared with **expi**, **expj**, and **expk** (the rotation matrices [complex exponentials]), and **exphi**, **expjh**, **expkh** (the hyperbolic matrices), and **expii**, **expjj**, and **expkk** (the exponential attenuation matrices). The nilpotent second order mixed double crosses are the exponentials of the non-diagonal terms in a cross-product operator: three of which ($k \times j \times$, $k \times i \times$, and $j \times i \times$) are **upper triangular matrix positions** [once diadically expanded] and the other three ($i \times j \times$, $i \times k \times$, and $j \times k \times$) **lower triangular matrix positions**.

Three of the six will be the focus for consideration, k -cross- j -cross ($k \times j \times$), j -cross- i -cross ($j \times i \times$), and k -cross- i -cross ($k \times i \times$) and their exponentials **expji**, **expki**, and **expkj**. These ($k \times j \times$, $j \times i \times$, and $k \times i \times$) are the **upper triangular matrix positions** once diadically expanded. For example, considering $\hat{k} \times \hat{j} \times$: for column basis components, j -hat is null immediately (from j -cross), i -hat is null from the k -cross after the first j -cross permutes it to k -hat, and so the only non-trivial operand of k -cross- j -cross ($k \times j \times$) is that operating on k -hat which returns j -hat

- $(\hat{k} \times)(\hat{j} \times)\hat{j} = (\hat{k} \times)\hat{j} \times \hat{j} = (\hat{k} \times)(0 \hat{j}) = 0(\hat{k} \times)\hat{j} = 0\hat{i}$
- $(\hat{k} \times)(\hat{j} \times)\hat{i} = (\hat{k} \times)(\hat{j} \times)\hat{i} = (\hat{k} \times)(\hat{j} \times \hat{i}) = (\hat{k} \times)(-1\hat{k}) = 0\hat{k}$
- $(\hat{k} \times)(\hat{j} \times)\hat{k} = (\hat{k} \times)(\hat{j} \times)\hat{k} = (\hat{k} \times)(\hat{j} \times \hat{k}) = (\hat{k} \times)\hat{i} = 1\hat{j}$

Upon diadic expansion mapping these vectors to columns, then, there is only one component in the matrix basis, the j -hat entry of the k -hat column. Hence

$$\hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Alternatively, since $\hat{k} \times$ and $\hat{j} \times$ have already been diadically expanded, the product of the two diadically expanded operators is

$$\hat{k} \times \hat{j} \times = (\hat{k} \times)(\hat{j} \times) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This and the other two, as a set, are

$$\hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{k} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also, the lower triangular mixed (nilpotent) double crosses are

$$\hat{i} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{j} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{i} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which are in the form of the transposes of the upper triangular nilpotent double crosses but without priming of the vector basis since these are obtained by diadic expansion. These imply the following three cross product identities discussed earlier

$$\hat{i} \times \hat{j} \times - \hat{j} \times \hat{i} \times = \hat{k} \times \quad \hat{j} \times \hat{k} \times - \hat{k} \times \hat{j} \times = \hat{i} \times \quad \hat{k} \times \hat{i} \times - \hat{i} \times \hat{k} \times = \hat{j} \times$$

The **minimum basis factorisation** of the third order upper 3by3 **nilpotent matrix** is the sum of the first two of the upper triangular nilpotent double crosses, $N = \hat{j} \times \hat{i} \times + \hat{k} \times \hat{j} \times$,

$$\hat{j} \times \hat{i} \times + \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = N_3$$

The square of the nilpotent matrix is the third above $N^2 = \hat{k} \times \hat{i} \times$ i.e.,

$$(\hat{j} \times \hat{i} \times + \hat{k} \times \hat{j} \times)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{k} \times \hat{i} \times$$

This also makes showing the nilpotent properties easy:

$$N = \hat{j} \times \hat{i} \times + \hat{k} \times \hat{j} \times = \hat{i} \hat{j}' + \hat{j} \hat{k}'$$

Squaring N for N^2

$$N^2 = (\hat{i} \hat{j}' + \hat{j} \hat{k}')^2 = (\hat{i} \hat{j}' + \hat{j} \hat{k}')(\hat{i} \hat{j}' + \hat{j} \hat{k}') = \hat{i} \hat{j}' \hat{i} \hat{j}' + \hat{j} \hat{k}' \hat{i} \hat{j}' + \hat{i} \hat{j}' \hat{j} \hat{k}' + \hat{j} \hat{k}' \hat{j} \hat{k}' = \hat{i} (0) \hat{j}' + \hat{j} (0) \hat{i}' + \hat{i} (1) \hat{k}' + \hat{j} (0) \hat{k}' = \hat{i} \hat{k}'$$

which is the diadic expansion of $\hat{k} \times \hat{i} \times$ and the upper right corner of the 3by3. The squaring of N hence has rank 1 nullity 2 increasing the nullity by 1 by the property of nilpotence of the two mixed double crosses

$$N^2 = \hat{i} \hat{k}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To third order power for the cube of N

$$N^3 = (\hat{i} \hat{j}' + \hat{j} \hat{k}')^3 = \hat{i} \hat{k}' (\hat{i} \hat{j}' + \hat{j} \hat{k}') = \hat{i} \hat{k}' \hat{i} \hat{j}' + \hat{i} \hat{k}' \hat{j} \hat{k}' = \hat{i} (0) \hat{j}' + \hat{i} (0) \hat{k}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is the zero matrix hence demonstrating nilpotence. This shows how a nilpotent matrix gets its properties, from combinations of k -cross, j -cross, and i -cross that terminate. The theory of nilpotence and Jordan forms then puts to work those combinations that go nowhere... combinations that lead to zeroes rather than infinite series. This then simplifies the mathematics and is the unique feature of the nilpotent operators.

For interest, comparing the above with the lower triangular entries and the equivalent lower triangular nilpotent matrix

$$\begin{aligned} \text{upper: } N_3 \hat{i} \hat{j}' + \hat{j} \hat{k}' &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \text{upper squared: } N_3^2 = \hat{i} \hat{k}' &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{the upper anti-diagonal 3 of the 6 mixed double crosses}) \\ \text{lower: } N_3 \hat{j} \hat{i}' + \hat{k} \hat{j}' &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \text{lower squared: } N_3^2 = \hat{k} \hat{i}' &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (\text{the lower anti-diagonal 3 of the 6 mixed double crosses}) \end{aligned}$$

The important feature of nilpotence is that it truncates power series expansions, and so simplifies mathematics in circumstances where it would otherwise become overwhelmingly complicated. Hence, nilpotence is used to simplify matters [between axes] and in this way is used to generate finite series as apposed to infinite series in the cross products, and hence form arithmetic and geometric sequences from matrix powers once the relevant eigenvalues are added by exponentiation.

Exponentiation of the nilpotent mixed double crosses is considered now. The mixed double cross exponentials, such as $\exp \mathbf{jk}$, $e^{\theta \hat{k} \times \hat{j} \times}(\cdot) = \exp(\theta \hat{k} \times \hat{j} \times)(\cdot)$, have been curiosities up to this point because of the nilpotence such as $\mathbf{k} \times \mathbf{i} \times = N^2$. These are much more than just a curiosity because these operators are what generates the intricacies of Jordan matrices and so is the theory used to obtain eigenvalues of non-diagonalizable matrices. This is also why the theory of Jordan matrices works similarly with either the upper diagonal entries or the lower diagonal entries. This provides an important **complement to harmonics and reflections**.

The exponentiation of the $\mathbf{k} \times \mathbf{j} \times$ (k -cross- j -cross) adds the unity diagonal elements, which makes the otherwise singular operator positive definite. Powers then provide a counting matrix where the power index is in the position stipulated by the mixed double cross, that is, in the jk' position in the case of $\mathbf{k} \times \mathbf{j} \times$. The mathematics of the powers is interesting since it generates the arithmetic sequence in the matrix basis for these powers.

Because k -cross- j -cross is null immediately after the first power (because of the nilpotence), all expansions of the exponential $\mathbf{k} \times \mathbf{j} \times$ (that of **expkj**) are at most first order in j -cross- k -cross

$$\begin{aligned} e^{\theta \hat{k} \times \hat{j} \times}(\cdot) &= \exp(\theta \hat{k} \times \hat{j} \times) \\ &= 1 + (\theta \hat{k} \times \hat{j} \times) + \frac{1}{2!}(\theta \hat{k} \times \hat{j} \times)^2 + \dots \\ &= 1 + \theta(\hat{k} \times \hat{j} \times) + \frac{1}{2!}\theta^2(\hat{k} \times \hat{j} \times)^2 + \dots \\ &= 1 + \theta(\hat{k} \times \hat{j} \times) + o(\hat{k} \times \hat{j} \times) \\ &= 1 + \theta(\hat{k} \times)(\hat{j} \times) + o(\hat{k} \times \hat{j} \times) \end{aligned}$$

where o is order notation here $o(\hat{k} \times \hat{j} \times)$ for powers in the mixed double cross $\hat{j} \times \hat{k} \times$. In each component direction

- column basis i -hat direction $e^{\theta \hat{k} \times \hat{j} \times}(\cdot)\hat{i} = \hat{i} + \theta(\hat{k} \times)(\hat{j} \times)\hat{i} + o(\hat{k} \times \hat{j} \times)\hat{i} = \hat{i}$
- column basis j -hat direction $e^{\theta \hat{k} \times \hat{j} \times}(\cdot)\hat{j} = \hat{j} + \theta(\hat{k} \times)(\hat{j} \times)\hat{j} + o(\hat{k} \times \hat{j} \times)\hat{j} = \hat{j}$
- column basis k -hat direction $e^{\theta \hat{k} \times \hat{j} \times}(\cdot)\hat{k} = \hat{k} + \theta(\hat{k} \times)(\hat{j} \times)\hat{k} + o(\hat{k} \times \hat{j} \times)\hat{j} = \theta\hat{j} + \hat{k}$

For a diadic expansion in the column basis, components become columns

$$\mathbf{expkj}\theta = e^{\theta \hat{k} \times \hat{j} \times}(\cdot) = \exp(\theta \hat{k} \times \hat{j} \times) = I + \theta \hat{k} \times \hat{j} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

or equivalently,

$$\mathbf{expkj}\theta = I + \theta \hat{k} \times \hat{j} \times = I + \theta(\hat{k} \times)(\hat{j} \times) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, the exponential has added the identity to the diadic expansion of the operator and so given the operator a diagonal making it full-rank. For **expji** and **expki**, the mathematics is similar as above. These are

$$\mathbf{expji}\theta = e^{\theta \hat{j} \times \hat{i} \times} = I + \theta \hat{j} \times \hat{i} \times = \begin{pmatrix} 1 & \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{expki}\theta = e^{\theta \hat{k} \times \hat{i} \times} = I + \theta \hat{k} \times \hat{i} \times = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The formulae $I + \theta \hat{k} \times \hat{j} \times$, $I + \theta \hat{j} \times \hat{i} \times$, and $I + \theta \hat{k} \times \hat{i} \times$ for **expkj**, **expji**, and **expki** are essentially **Euler forms** for the basis exponentials with basis nilpotent cross-products rather than cyclic properties of $\mathbf{k} \times$, $\mathbf{j} \times$, and $\mathbf{i} \times$. Because of the nilpotence, the cyclic properties of the harmonics (**coskj**, **cosji**, **coski**, **sinkj**, **sinji**, **sinji**, and **sinki**) essentially breakdown and so form identity matrices of the small angle approximations for cosine and sine

$$\mathbf{coskj}\theta = \mathbf{cosji}\theta = \mathbf{coski}\theta = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{sinkj}\theta = \mathbf{sinji}\theta = \mathbf{sinki}\theta = \theta I = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

Therefore these diadically expanded matrix operators of basis exponentials can alternatively be expressed in an Euler type form

$$\mathbf{expkj}\theta = \mathbf{coskj}\theta + (\mathbf{sinkj}\theta)\hat{k} \times \hat{j} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } \mathbf{coskj}\theta = I, \mathbf{sinkj}\theta = \theta I = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}, \text{ and } \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{expji}\theta = \mathbf{cosji}\theta + (\mathbf{sinji}\theta)\hat{j} \times \hat{i} \times = \begin{pmatrix} 1 & \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } \mathbf{cosji}\theta = I, \mathbf{sinji}\theta = \theta I = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}, \text{ and } \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & \theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{expki}\theta = \mathbf{coski}\theta + (\mathbf{sinki}\theta)\hat{k} \times \hat{i} \times = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } \mathbf{coski}\theta = I, \mathbf{sinki}\theta = \theta I = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}, \text{ and } \hat{k} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & \theta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So therefore, since there are **Euler type expressions**, there is also the **de-Moivre's theorem** for powers. So for **de-Moivre's theorem** applied to the nilpotent basis function, raising **expkj** θ to the n th power for **(expkj) θ** ^{n} and binomially expanding

$$e^{\theta \hat{k} \times \hat{j} \times}(\cdot) = \left(\exp(\theta \hat{k} \times \hat{j} \times) \right)^n = \left(I + \theta \hat{k} \times \hat{j} \times \right)^n = I^n + n\theta \hat{k} \times \hat{j} \times + o(\hat{k} \times \hat{j} \times) = I + n\theta \hat{k} \times \hat{j} \times := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, **de-Moivre's theorem** for the basis exponential **expkj** θ of the complex-like number $\hat{k} \times \hat{j} \times$ is

$$\left(\exp(\theta \hat{k} \times \hat{j} \times) \right)^n = 1 + n\theta \hat{k} \times \hat{j} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n\theta \\ 0 & 0 & 1 \end{pmatrix}$$

i.e.,

$$(\text{expkj}\theta)^n = \left(\cos\mathbf{j}\theta + (\sin\mathbf{j}\theta)\hat{k} \times \hat{j} \times \right)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}^n = \cos\mathbf{j}n\theta + (\sin\mathbf{j}n\theta)\hat{k} \times \hat{j} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly

$$(\text{expji}\theta)^n = \left(\cos\mathbf{j}\theta + (\sin\mathbf{j}\theta)\hat{j} \times \hat{i} \times \right)^n = \begin{pmatrix} 1 & \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \cos\mathbf{j}n\theta + (\sin\mathbf{j}n\theta)\hat{j} \times \hat{i} \times = \begin{pmatrix} 1 & n\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\text{expki}\theta)^n = \left(\cos\mathbf{k}\theta + (\sin\mathbf{k}\theta)\hat{k} \times \hat{i} \times \right)^n = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \cos\mathbf{k}n\theta + (\sin\mathbf{k}n\theta)\hat{k} \times \hat{i} \times = \begin{pmatrix} 1 & 0 & n\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as well as for the three lower triangular mixed double cross basis exponentials.

In comparison with the rotation matrix and attenuation matrices, this mathematics shows how the nilpotent entries are uniquely useful. This usefulness is hence demonstrated as the fact that they truncate the power series when in exponential form. That is why they are indispensable and core to the theory of Jordan matrices and the theory of otherwise non-diagonalisable matrices. The first five powers of the curiosity of the k -cross- j -cross exponential **expkj**

$$\left(e^{\theta \hat{k} \times \hat{j} \times} \right)^1, \left(e^{\theta \hat{k} \times \hat{j} \times} \right)^2, \left(e^{\theta \hat{k} \times \hat{j} \times} \right)^3, \left(e^{\theta \hat{k} \times \hat{j} \times} \right)^4, \left(e^{\theta \hat{k} \times \hat{j} \times} \right)^5$$

are

$$\text{expkj}\theta \quad (\text{expkj}\theta)^2 \quad (\text{expkj}\theta)^3 \quad (\text{expkj}\theta)^4 \quad (\text{expkj}\theta)^5$$

$$\textbf{k} \times \textbf{j} \times \text{ exponential,} \quad (\text{expkj}\theta)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1\theta \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\theta \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3\theta \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4\theta \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly for the other two upper diagonal nilpotent basis exponentials **expji** and **expki**

$$\text{expji}\theta \quad (\text{expji}\theta)^2 \quad (\text{expji}\theta)^3 \quad (\text{expji}\theta)^4 \quad (\text{expji}\theta)^5$$

$$\textbf{j} \times \textbf{i} \times \text{ exponential:} \quad (\text{expji}\theta)^n = \begin{pmatrix} 1 & 1\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{expki}\theta \quad (\text{expki}\theta)^2 \quad (\text{expki}\theta)^3 \quad (\text{expki}\theta)^4 \quad (\text{expki}\theta)^5$$

$$\textbf{k} \times \textbf{i} \times \text{ exponential,} \quad (\text{expki}\theta)^n = \begin{pmatrix} 1 & 0 & 1\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 4\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 5\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which shows that the kk' for the $\textbf{k} \times \textbf{i} \times$ exponential **expki** is in arithmetic progression noting that these would be the zero matrix after the second power if the identity had not been added from exponentiation. Note that reversing the order of the basis -crosses that are exponentiated (before diadically expanding) changes the positions from, as here, the **upper triangular positions** to the equivalent **lower triangular positions** [and so why Jordan forms can be implemented with either]. The nilpotence instead means that powers are

in single term arithmetic progression rather than infinite sine and cosine series that sum into harmonics. The mixed basis-double-crosses in the two nilpotent Jordan positions can be combined in the exponential of their sum

$$\exp \mathbf{kj}(a) \exp \mathbf{ji}(b) = (I + a \hat{k} \times \hat{j} \times) (I + b \hat{j} \times \hat{i} \times) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

Note that when writing this page out, I was concentrating on the powers so this document had two equations that do not hold good which I should have realised but doesn't matter because it is a basis for some further theories regarding **(1)** when diadic expansions hold for conjugated functions of basis operators and **(2)** also explains why the cross operators commute with plus '+' but not in multiplication after exponentiation (which is a result in Jordan matrices which can be extended to that for diadically expanded operators). This is

$$\begin{aligned} \exp(a \hat{k} \times \hat{j} \times + b \hat{j} \times \hat{i} \times) &= I + (a \hat{k} \times \hat{j} \times + b \hat{j} \times \hat{i} \times) + \frac{1}{2} (a \hat{k} \times \hat{j} \times + b \hat{j} \times \hat{i} \times)^2 \\ &= I + \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & b & \frac{1}{2}ab \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and so $\exp(a \hat{k} \times \hat{j} \times + b \hat{j} \times \hat{i} \times) \neq \exp \mathbf{kj}(a) \exp \mathbf{ji}(b)$ which is a refresher for Jordan forms since the first row in an exponential of a Jordan form is a Poisson series, here to order 2. Normally in solving a Jordan eigenvalue problem a and b are time t so this matrix would be

$$\exp(t(\hat{k} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

which just goes to show that this maths in terms of cross products as complex numbers [basis rotation operators] works exactly as it should with known mathematics, mathematics which is the crescendo of UNSW second year linear algebra. Note that this is the important and crescendo result for $\exp(t J_3(0))$ which further demonstrates that the mathematics of basis functions appended with $\times$ as complex number-like rotation operators (here the mixed, and hence nilpotent, double crosses) works perfectly in conjunction with known linear algebra. Note that

$$\exp(t J_3(\lambda)) = \exp(\lambda t + t J_3(0)) = \exp(\lambda t) \exp(t J_3(0)) = \exp(\lambda t) \exp(t(\hat{k} \times \hat{j} \times + \hat{j} \times \hat{i} \times)) = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

where in this instance the exponentials of J and its argument λ do break into products. The mathematics of solving Jordan systems can be complicated so this is another possible area where the theory of basis operators (and their diadic expansions) can likely improve understanding and make the mathematics more intuitive and easier to apply.

Powers of the two basis exponentials (and not the exponential of the two component operators) result in entries being the first terms in binomial series

$$\exp \mathbf{kj}(a) \exp \mathbf{ji}(b) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2a & ab \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3a & 3ab \\ 0 & 1 & 3b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4a & 6ab \\ 0 & 1 & 4b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5a & 10ab \\ 0 & 1 & 5b \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\exp \mathbf{kj}(a) \exp \mathbf{ji}(b) \exp \mathbf{ki}(c) = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} (I + c \hat{k} \times \hat{i} \times) = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

for the series of powers

$$\mathbf{expkj}(a)\mathbf{expji}(b)\mathbf{expki}(c) \quad \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2a & ab+2c \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3a & 3ab+3c \\ 0 & 1 & 3b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4a & 6ab+4c \\ 0 & 1 & 4b \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5a & 10ab+5c \\ 0 & 1 & 5b \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonal entries can also be scaled which places powers of ascending order along rows. The mixed double crosses introduces the nilpotence and then the exponentiation removes that nilpotence [by adding the eigenvalues] to form the geometric series. This series is in contrast to harmonic and hyperbolic for the other basis exponentials, i.e., harmonics for **expi**, **expj**, and **expk**, exponential attenuation for **expii**, **expjj**, and **expkk**, and hyperbolics for **expji**, **expij**, and **expik** which I am comparing and contrasting here.

Note that $\exp(a\hat{k} \times \hat{j} \times + b\hat{j} \times \hat{i} \times + c\hat{k} \times \hat{i} \times) \neq \mathbf{expkj}(a)\mathbf{expji}(b)\mathbf{expki}(c)$ and rather

$$\begin{aligned} \exp(a\hat{k} \times \hat{j} \times + b\hat{j} \times \hat{i} \times + c\hat{k} \times \hat{i} \times) &= I + (a\hat{k} \times \hat{j} \times + b\hat{j} \times \hat{i} \times + c\hat{k} \times \hat{i} \times) + \frac{1}{2}(a\hat{k} \times \hat{j} \times + b\hat{j} \times \hat{i} \times + c\hat{k} \times \hat{i} \times)^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & b & c \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & b & c + \frac{1}{2}ab \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

nb: because a product of diadically expanded operators equals the diadic expansion of the product, I had assumed whilst solving on the fly that this would also hold for exponentials which it does not which is good to know and explains the issue of non-commuting basis exponentials despite the sum of the cross-product indices appearing to commute. When a, b , and c are time t this is

$$\exp(t(\hat{k} \times \hat{j} \times + b\hat{j} \times \hat{i} \times + c\hat{k} \times \hat{i} \times)) = \begin{pmatrix} 1 & t & t + \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Purely for interest, these powers and resulting progression for the mixed double-cross exponential **ciskj** can be compared with **de-Moivre's theorem** for those powers for the single basis-cross exponential **cisk** $(e^{\theta\hat{k} \times \hat{j}})^1, (e^{\theta\hat{k} \times})^2, (e^{\theta\hat{k} \times})^3, (e^{\theta\hat{k} \times})^4, (e^{\theta\hat{k} \times})^5$

$$\begin{pmatrix} \cos 1\theta & -\sin 1\theta & 0 \\ \sin 1\theta & \cos 1\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos 3\theta & -\sin 3\theta & 0 \\ \sin 3\theta & \cos 3\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos 4\theta & -\sin 4\theta & 0 \\ \sin 4\theta & \cos 4\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos 5\theta & -\sin 5\theta & 0 \\ \sin 5\theta & \cos 5\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which shows clearly that **expkj** powers are a form of **de-Moivre's theorem** when the component harmonics (complex exponentials) have nilpotence and hence form geometric series rather than harmonic and hyperbolic series. Note further that it is the first order k -cross, the quartic root of unity for the xy -plane, that puts the sines on the anti-diagonals which maintains the harmonics with [de-Moivre] powers.

Also comparing with the series in the k -hat double cross exponential **expkk** $(e^{\theta\hat{k} \times \hat{k} \times})^1, (e^{\theta\hat{k} \times \hat{k} \times})^2, (e^{\theta\hat{k} \times \hat{k} \times})^3, (e^{\theta\hat{k} \times \hat{k} \times})^4, (e^{\theta\hat{k} \times \hat{k} \times})^5$

$$\begin{aligned} &\mathbf{expkk} \theta \quad (\mathbf{expkk} \theta)^2 \quad (\mathbf{expkk} \theta)^3 \quad (\mathbf{expkk} \theta)^4 \quad (\mathbf{expkk} \theta)^5 \\ k \times k \times \text{ exponential, } (\mathbf{expkk} \theta)^n &= \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{-2\theta} & 0 & 0 \\ 0 & e^{-2\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{-3\theta} & 0 & 0 \\ 0 & e^{-3\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{-4\theta} & 0 & 0 \\ 0 & e^{-4\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{-5\theta} & 0 & 0 \\ 0 & e^{-5\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which is definitely interesting for comparison purposes.

Theorem: Every basis exponential has a **Euler form** by decomposition into even and odd components. This is by virtue of the power factor expansion of the relevant exponential cross-product or series of cross-products.

Corollary: Every basis exponential has a generalized form of **de-Moivre's theorem** for powers. This is by virtue of raising basis exponentials to powers.

Summarizing as **Euler forms** and powers for (1) cyclic (harmonic), (2) attenuation (exponential), (3) hyperbolic, and (4) nilpotent (progression) **de-Moivre's theorem(s)**:

(1) cyclic (harmonic)

$$(\mathbf{expk}\theta)^n = (\mathbf{cosk}\theta + (\mathbf{sink}\theta)\hat{k} \times)^n = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \mathbf{cosk}n\theta + (\mathbf{sink}n\theta)\hat{k} \times = \begin{pmatrix} \cos n\theta & -\sin n\theta & 0 \\ \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\mathbf{expj}\theta)^n = (\mathbf{cosj}\theta + (\mathbf{sinj}\theta)\hat{j} \times)^n = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}^n = \mathbf{cosj}n\theta + (\mathbf{sinj}n\theta)\hat{j} \times = \begin{pmatrix} \cos n\theta & 0 & \sin n\theta \\ 0 & 1 & 0 \\ -\sin n\theta & 0 & \cos n\theta \end{pmatrix}$$

$$(\mathbf{expi}\theta)^n = (\mathbf{cosi}\theta + (\mathbf{sini}\theta)\hat{i} \times)^n = \begin{pmatrix} 1 & 0 & \theta \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}^n = \mathbf{cosi}n\theta + (\mathbf{sini}n\theta)\hat{i} \times = \begin{pmatrix} 1 & 0 & \theta \\ 0 & \cos n\theta & -\sin n\theta \\ 0 & \sin n\theta & \cos n\theta \end{pmatrix}$$

(2) attenuation (exponential)

$$(\mathbf{expkk}\theta)^n = (\mathbf{coskk}\theta + (\mathbf{sinkk}\theta)\hat{k} \times \hat{k} \times)^n = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \mathbf{coskk}n\theta + (\mathbf{sinkk}n\theta)\hat{k} \times \hat{k} \times = \begin{pmatrix} e^{-n\theta} & 0 & 0 \\ 0 & e^{-n\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\mathbf{expjj}\theta)^n = (\mathbf{cosjj}\theta + (\mathbf{sinjj}\theta)\hat{j} \times \hat{j} \times)^n = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix}^n = \mathbf{cosjj}n\theta + (\mathbf{sinjj}n\theta)\hat{j} \times \hat{j} \times = \begin{pmatrix} e^{-n\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-n\theta} \end{pmatrix}$$

$$(\mathbf{expii}\theta)^n = (\mathbf{cosii}\theta + (\mathbf{sinii}\theta)\hat{i} \times \hat{i} \times)^n = \begin{pmatrix} 1 & 0 & \theta \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix}^n = \mathbf{cosii}n\theta + (\mathbf{sinii}n\theta)\hat{i} \times \hat{i} \times = \begin{pmatrix} 1 & 0 & \theta \\ 0 & e^{-n\theta} & 0 \\ 0 & 0 & e^{-n\theta} \end{pmatrix}$$

(3) hyperbolic

$$(\mathbf{expkh}\theta)^n = (\mathbf{coshk}\theta + (\mathbf{sinhk}\theta)(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times))^n = \mathbf{coshk}(n\theta) + (\mathbf{sinhk}(n\theta))(\hat{i} \times \hat{j} \times + \hat{j} \times \hat{i} \times) = \begin{pmatrix} \cosh n\theta & \sinh n\theta & 0 \\ \sinh n\theta & \cosh n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\mathbf{expjh}\theta)^n = (\mathbf{coshj}\theta + (\mathbf{sinjh}\theta)(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times))^n = \mathbf{coshj}(n\theta) + (\mathbf{sinjh}(n\theta))(\hat{k} \times \hat{i} \times + \hat{i} \times \hat{k} \times) = \begin{pmatrix} \cosh n\theta & 0 & \sinh n\theta \\ 0 & 1 & 0 \\ \sinh n\theta & 0 & \cosh n\theta \end{pmatrix}$$

$$(\mathbf{expih}\theta)^n = (\mathbf{coshi}\theta + (\mathbf{sinhi}\theta)(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times))^n = \mathbf{coshi}(n\theta) + (\mathbf{sinhi}(n\theta))(\hat{j} \times \hat{k} \times + \hat{k} \times \hat{j} \times) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh n\theta & \sinh n\theta \\ 0 & \sinh n\theta & \cosh n\theta \end{pmatrix}$$

(4) nilpotent (progression)

$$(\mathbf{expkj}\theta)^n = (\mathbf{coskj}\theta + (\mathbf{sinkj}\theta)\hat{k} \times \hat{j} \times)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}^n = \mathbf{coskj}n\theta + (\mathbf{sinkj}n\theta)\hat{k} \times \hat{j} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\mathbf{expji}\theta)^n = (\mathbf{cosji}\theta + (\mathbf{sinji}\theta)\hat{j} \times \hat{i} \times)^n = \begin{pmatrix} 1 & \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \mathbf{cosji}n\theta + (\mathbf{sinji}n\theta)\hat{j} \times \hat{i} \times = \begin{pmatrix} 1 & n\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\mathbf{expki}\theta)^n = \left(\coski\theta + (\mathbf{sinki}\theta) \hat{k} \times \hat{i} \times \right)^n = \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \coskin\theta + (\mathbf{sinkin}\theta) \hat{k} \times \hat{i} \times = \begin{pmatrix} 1 & 0 & n\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

All of these formulae also hold for the cis-scaled bases, those of polar cylindrical ($\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{z}$ -hat) and polar spherical ($\hat{r}$ -hat, $\hat{\theta}$ -hat, and $\hat{\phi}$ -hat) bases.

So it first appears that the interactions between the complex numbers on each of the orthogonal planes that zero and appear to lead nowhere are not as interesting as the complex numbers on each of the planes ($\hat{k} \times$, $\hat{j} \times$, $\hat{i} \times$). It is then seen that these generate series from simplifications when expanded. It is then seen further that these are the necessary features of Jordan matrices and solving progressively higher orders of differential equations that are signified by non-diagonalizable matrices. This shows the importance of the interaction and hence quaternion properties of the single component basis cross products when they interact between the orthogonal axes (that is, when they operate for more than one of the three principal rotation axes).

6. Quaternion Triple Cross and Basis Exponential

The third order quaternion-like crosses are considered now. These are the single component third order cross products consisting of $\hat{k} \times$, $\hat{j} \times$, $\hat{i} \times$ in the six possible permutations. These operators return a specific basis-hat (single component basis-vector) after being applied to that basis-hat. This means that they are identity for that basis-hat, one positive and one negative. The three that are positive are consequently idempotent operators. For a quaternion these were the positive equivalents to $ijk = -1$ and why I have called the mixed triple crosses quaternions despite actually being a single term in a vector triple product. This is actually $\hat{i} \times \hat{j} \times \hat{k} \times (\cdot) = -1(\cdot)$ for $\hat{j}$ -hat ($\hat{j}$) and is null for $\hat{k}$ -hat and $\hat{i}$ -hat. Since the component basis-crosses do not commute and hence changing the order [in this particular instance] changes the sign, that is [if these were complex numbers], $kij = +1$, or more correctly, $\hat{k} \times \hat{j} \times \hat{i} \times (\cdot) = +1(\cdot)$ for $\hat{j}$ -hat, $\hat{j}$ (and also null for $\hat{k}$ -hat and $\hat{i}$ -hat). Interesting though, and as an ongoing them, although the individual basis-crosses don't commute, when taken together as a single operator, it commutes with others of the same type so that any function of $\hat{k} \times \hat{j} \times \hat{i} \times (\cdot) = +1(\cdot)$ will commute with any other function of $\hat{k} \times \hat{j} \times \hat{i} \times (\cdot) = +1(\cdot)$ which may prove useful at some stage when difficulties are encountered from terms otherwise not commuting.

As basis operators, the quaternion-like operator [in the mutually perpendicular basis-crosses] are kji for $\hat{j}$ -hat, ikj for $\hat{k}$ -hat, and jik for $\hat{i}$ -hat, that is, $\hat{k} \times \hat{j} \times \hat{i} \times$ for $\hat{j}$ -hat, $\hat{i} \times \hat{k} \times \hat{j} \times$ for $\hat{k}$ -hat, and $\hat{j} \times \hat{i} \times \hat{k} \times$ for $\hat{i}$ -hat. These are the three cyclic permutations of kji (that is, the cyclic permutations of ijk in reverse, those of ijk are the negative identities) and are highlighted in the third order curl tables of the first k -cross document I wrote.

In the required full notation of k -cross, j -cross, and i -cross, these are the quaternion identity operators for each basis vector, the **positive quaternion triples crosses**

$\hat{k} \times \hat{j} \times \hat{i} \times = +1$ (kji) for $\hat{j}$ -hat, $\hat{j}$ (the positive $\hat{j}$ -hat quaternion, noting that $\hat{j}$ -hat is in the middle) and null for $\hat{k}$ -hat and $\hat{i}$ -hat

$\hat{j} \times \hat{i} \times \hat{k} \times = +1$ (ikj) for $\hat{k}$ -hat, $\hat{k}$ (the positive $\hat{k}$ -hat quaternion, noting that $\hat{k}$ -hat is in the middle) and null for $\hat{i}$ -hat and $\hat{j}$ -hat

$\hat{i} \times \hat{k} \times \hat{j} \times = +1$ (jik) for $\hat{i}$ -hat, $\hat{i}$ (the positive $\hat{i}$ -hat quaternion, noting that $\hat{i}$ -hat is in the middle) and null for $\hat{j}$ -hat and $\hat{k}$ -hat

i.e., the cyclic permutations of the three factors of $\hat{k} \times \hat{j} \times \hat{i} \times$. With the cyclic permutations in the reversed order, $\hat{i} \times \hat{j} \times \hat{k} \times$, the negative $\hat{j}$ -hat, $\hat{k}$ -hat, and $\hat{i}$ -hat quaternions (the **negative quaternion triples crosses**) are

$\hat{i} \times \hat{j} \times \hat{k} \times = -1$ (kji) for $\hat{j}$ -hat, $\hat{j}$ (the negative $\hat{j}$ -hat quaternion, noting that $\hat{j}$ -hat is in the middle) and null for $\hat{k}$ -hat and $\hat{i}$ -hat

$\hat{j} \times \hat{k} \times \hat{i} \times = -1$ (ikj) for $\hat{k}$ -hat, $\hat{k}$ (the negative $\hat{k}$ -hat quaternion, noting that $\hat{k}$ -hat is in the middle) and null for $\hat{i}$ -hat and $\hat{j}$ -hat

$\hat{k} \times \hat{i} \times \hat{j} \times = -1$ (jik) for $\hat{i}$ -hat, $\hat{i}$ (the negative $\hat{i}$ -hat quaternion, noting that $\hat{i}$ -hat is in the middle) and null for $\hat{j}$ -hat and $\hat{k}$ -hat

In the curl tables (tables for cross products as complex numbers) for third order,

(1) for the kji permutations, $+\hat{j}$ shows up in the $\hat{j}$ -hat column of $\hat{k} \times \hat{j} \times \hat{i} \times$, $+\hat{i}$ in the $\hat{i}$ -hat column of $\hat{j} \times \hat{i} \times \hat{k} \times$, and $+\hat{k}$ in the $\hat{k}$ -hat column of $\hat{i} \times \hat{j} \times \hat{k} \times$

and similarly

(2) for the ijk permutations, $-\hat{j}$ shows up in the j -hat column of $\hat{i} \times \hat{j} \times \hat{k} \times$, $-\hat{i}$ in the i -hat column $\hat{k} \times \hat{i} \times \hat{j} \times$, and $-\hat{k}$ in the k -hat column of $\hat{j} \times \hat{k} \times \hat{i} \times$

Because each of the six have only one non-null component, it makes for simple diadic expansion. For the **positive triple cross quaternion**, the components are

$$\begin{aligned} \hat{k} \times \hat{j} \times \hat{i} \times &= +1 & \Rightarrow & (\hat{k} \times \hat{j} \times \hat{i} \times) \hat{i} = 0\hat{i} & (\hat{k} \times \hat{j} \times \hat{i} \times) \hat{j} &= +1\hat{j} & (\hat{k} \times \hat{j} \times \hat{i} \times) \hat{k} &= 0\hat{k} \\ \hat{j} \times \hat{i} \times \hat{k} \times &= +1 & \Rightarrow & (\hat{j} \times \hat{i} \times \hat{k} \times) \hat{i} &= +1\hat{i} & (\hat{j} \times \hat{i} \times \hat{k} \times) \hat{j} &= 0\hat{j} & (\hat{j} \times \hat{i} \times \hat{k} \times) \hat{k} &= 0\hat{k} \\ \hat{i} \times \hat{k} \times \hat{j} \times &= +1 & \Rightarrow & (\hat{i} \times \hat{k} \times \hat{j} \times) \hat{i} &= 0\hat{i} & (\hat{i} \times \hat{k} \times \hat{j} \times) \hat{j} &= 0\hat{j} & (\hat{i} \times \hat{k} \times \hat{j} \times) \hat{k} &= +1\hat{k} \end{aligned}$$

Mapping components to rows, the diadic expansion of the cyclic permutations of kji , $\hat{k} \times \hat{j} \times \hat{i} \times$, $\hat{j} \times \hat{i} \times \hat{k} \times$, and, $\hat{i} \times \hat{k} \times \hat{j} \times$ are

$$\hat{k} \times \hat{j} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{j} \times \hat{i} \times \hat{k} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{i} \times \hat{k} \times \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As an alternative to diadic expansion, multiplying out the diadically expanded operators of $\hat{k} \times$, $\hat{j} \times$, and $\hat{i} \times$ in the relevant positive sequences ($kji = +1$)

$$\begin{aligned} \hat{k} \times \hat{j} \times \hat{i} \times &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \hat{j} \times \hat{i} \times \hat{k} \times &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{i} \times \hat{k} \times \hat{j} \times &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Showing, as an ongoing theme, that it all works as it should and the diadic expansions are correct.

Seems somewhat trivial but it is worth mentioning that the sum of the cyclic permutations of jik (that is, $\hat{j} \times \hat{i} \times \hat{k} \times + \hat{k} \times \hat{j} \times \hat{i} \times + \hat{i} \times \hat{k} \times \hat{j} \times$) form a **basis cross product factorization** of the **identity matrix** $\hat{k} \times \hat{j} \times \hat{i} \times + \hat{j} \times \hat{i} \times \hat{k} \times + \hat{i} \times \hat{k} \times \hat{j} \times = I$. This seems quite reasonable given that they are the idempotent operators for each of the basis vectors. Their sum is I , that is, the identity matrix as an operator for a column position vector

$$\hat{j} \times \hat{i} \times \hat{k} \times + \hat{k} \times \hat{j} \times \hat{i} \times + \hat{i} \times \hat{k} \times \hat{j} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

which is kind of trivial, and kind of not trivial. It is interesting that the identity matrix can be considered the cyclic permutation sum of quaternion complex numbers (heterogenous basis triple cross products).

Also, another **basis cross product factorisation** of the identity consists of pairs of double crosses (quadruple crosses) since each double cross has nullity 1 and their products nullity 2

$$(\hat{k} \times)^2 (\hat{i} \times)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\hat{i} \times)^2 (\hat{j} \times)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\hat{j} \times)^2 (\hat{k} \times)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(\hat{k} \times)^2 (\hat{i} \times)^2 + (\hat{i} \times)^2 (\hat{j} \times)^2 + (\hat{j} \times)^2 (\hat{k} \times)^2 = I$. Note for example that

$$(\hat{k} \times)^2 (\hat{i} \times)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and similarly for

$$(\hat{i} \times)^2 (\hat{j} \times)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (\hat{j} \times)^2 (\hat{k} \times)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which can also be obtained from a change of base. This also suggests cross-product/complex-number/quaternion identities. Because $(\hat{k} \times)^2 (\hat{i} \times)^2$, $(\hat{i} \times)^2 (\hat{j} \times)^2$, and $(\hat{j} \times)^2 (\hat{k} \times)^2$ have the same diadically expanded forms as $\hat{k} \times \hat{j} \times \hat{i} \times$, $\hat{i} \times \hat{k} \times \hat{j} \times$, and $\hat{j} \times \hat{i} \times \hat{k} \times$ respectively, they are equivalent. These particular identities are therefore

$$(1) \hat{k} \times \hat{j} \times \hat{i} \times = (\hat{k} \times)^2 (\hat{i} \times)^2$$

$$(2) \hat{i} \times \hat{k} \times \hat{j} \times = (\hat{i} \times)^2 (\hat{j} \times)^2$$

$$(3) \hat{j} \times \hat{i} \times \hat{k} \times = (\hat{j} \times)^2 (\hat{k} \times)^2$$

A good name for these is **idempotent identities** for the positive triple-cross quaternions. Once one is identified, the others in a similarity group can be identified and obtained by change of base between the principal axes. These identities also hold for polar cylindrical and polar spherical bases. There are many more. These essentially arise from group properties as observed as commutator sequences on Rubiks cubes (Rubiks quaternions). I envisage that these may be useful for simplifying non-commutative algebra. Some more idempotent identities arise from matching

$$\hat{i} \times (\hat{j} \times)^2 \hat{i} \times = \hat{k} \times (\hat{j} \times)^2 \hat{k} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{i} \times (\hat{k} \times)^2 \hat{i} \times = \hat{j} \times (\hat{k} \times)^2 \hat{j} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \hat{j} \times (\hat{i} \times)^2 \hat{j} \times = \hat{k} \times (\hat{i} \times)^2 \hat{k} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with either the nilpotent quaternion triple crosses or the paired double crosses. For example,

$$\hat{i} \times (\hat{j} \times)^2 \hat{i} \times = (\hat{k} \times)^2 (\hat{i} \times)^2$$

$$\hat{i} \times (\hat{k} \times)^2 \hat{i} \times = (\hat{i} \times)^2 (\hat{j} \times)^2$$

$$\hat{j} \times (\hat{i} \times)^2 \hat{j} \times = (\hat{j} \times)^2 (\hat{k} \times)^2$$

$$\hat{k} \times (\hat{j} \times)^2 \hat{k} \times = (\hat{k} \times)^2 (\hat{i} \times)^2$$

$$\hat{j} \times (\hat{k} \times)^2 \hat{j} \times = (\hat{i} \times)^2 (\hat{j} \times)^2$$

$$\hat{k} \times (\hat{i} \times)^2 \hat{k} \times = (\hat{j} \times)^2 (\hat{k} \times)^2$$

$$\hat{i} \times (\hat{j} \times)^2 \hat{i} \times = \hat{k} \times \hat{j} \times \hat{i} \times$$

$$\hat{i} \times (\hat{k} \times)^2 \hat{i} \times = \hat{i} \times \hat{k} \times \hat{j} \times$$

$$\hat{j} \times (\hat{i} \times)^2 \hat{j} \times = \hat{j} \times \hat{i} \times \hat{k} \times$$

$$\hat{k} \times (\hat{j} \times)^2 \hat{k} \times = \hat{k} \times \hat{j} \times \hat{i} \times$$

$$\hat{j} \times (\hat{k} \times)^2 \hat{j} \times = \hat{i} \times \hat{k} \times \hat{j} \times$$

$$\hat{k} \times (\hat{i} \times)^2 \hat{k} \times = \hat{j} \times \hat{i} \times \hat{k} \times$$

Useful or not for application, these are worthy of enumeration purely for academic interest. They (the identities) look good as well. These types of identities may one day provide a means for solving god's number for a Rubiks quaternion with an analytical formula. For the **negative triple cross quaternions** (the reversed cyclic permutations, those of $ijk = -1$ rather than $kji = +1$) components for diadic expansion are

$$\begin{array}{llll} \hat{i} \times \hat{j} \times \hat{k} \times = -1 & \Rightarrow & (\hat{i} \times \hat{j} \times \hat{k} \times) \hat{i} = 0\hat{i} & (\hat{i} \times \hat{j} \times \hat{k} \times) \hat{j} = -1\hat{j} & (\hat{i} \times \hat{j} \times \hat{k} \times) \hat{k} = 0\hat{k} \\ \hat{j} \times \hat{k} \times \hat{i} \times = -1 & \Rightarrow & (\hat{j} \times \hat{k} \times \hat{i} \times) \hat{j} = 0\hat{j} & (\hat{j} \times \hat{k} \times \hat{i} \times) \hat{k} = 0\hat{k} & (\hat{j} \times \hat{k} \times \hat{i} \times) \hat{i} = -1\hat{i} \\ \hat{k} \times \hat{i} \times \hat{j} \times = -1 & \Rightarrow & (\hat{k} \times \hat{i} \times \hat{j} \times) \hat{k} = -1\hat{k} & (\hat{k} \times \hat{i} \times \hat{j} \times) \hat{i} = 0\hat{i} & (\hat{k} \times \hat{i} \times \hat{j} \times) \hat{j} = 0\hat{j} \end{array}$$

and with components mapped to columns

$$\hat{i} \times \hat{j} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{j} \times \hat{k} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \hat{k} \times \hat{i} \times \hat{j} \times = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or alternatively for the negative quaternion triple-crosses with the relevant negative sequences ($ijk = -1$) of the individual crosses

$$\begin{array}{l} \hat{i} \times \hat{j} \times \hat{k} \times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hat{j} \times \hat{k} \times \hat{i} \times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ \hat{k} \times \hat{i} \times \hat{j} \times = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

which are the same as the diadic expansions.

Summing the three negative triple-cross quaternions result in the negative of the identity matrix, i.e., a **basis cross product factorisation** of the identity $\hat{k} \times \hat{i} \times \hat{j} \times + \hat{i} \times \hat{j} \times \hat{k} \times + \hat{j} \times \hat{k} \times \hat{i} \times = -I$

$$\hat{k} \times \hat{i} \times \hat{j} \times + \hat{i} \times \hat{j} \times \hat{k} \times + \hat{j} \times \hat{k} \times \hat{i} \times = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Because of the negative sign, these aren't actually idempotent (despite the similarity with the three that are) because they osculate in sign for powers (which make them periodic). The equivalent **identities** for the negative triple cross quaternions are

$$(1) \hat{i} \times \hat{j} \times \hat{k} \times = -(\hat{i} \times)^2 (\hat{k} \times)^2$$

$$(2) \hat{j} \times \hat{k} \times \hat{i} \times = -(\hat{j} \times)^2 (\hat{i} \times)^2$$

$$(3) \hat{k} \times \hat{i} \times \hat{j} \times = -(\hat{k} \times)^2 (\hat{j} \times)^2$$

The exponential of the mixed triple cross quaternions, **expkji**, **expikj**, and **expjik** (or **ciskji**, **cisikj**, and **cisjik** with suggested pronunciation **sis-kay-jay-eye**, **sis-eye-kay-jay**, and **sis-jay-eye-kay**) work similarly to the double crosses (cis in some sense loses its meaning when not harmonic so the suggested pronunciation is probably better **exp-kay-jay-eye**, **exp-eye-kay-jay**, and **exp-jay-eye-kay**).

Because of the similarity (of the heterogonous triple crosses) with the (homogenous) double crosses, rather than harmonic (single crosses) or the nilpotent (heterogeneous double crosses), it is self-evident that these exponentials will be hyperbolic-like (and so exponential damp or magnify). The difference here between the homogenous double crosses and heterogeneous quaternion triple crosses, is that the quintessential feature of the square being neg-one holding for two axes for the homogenous double crosses (squared complex numbers) is rather for one axis rather than two because it is the monomial power (idempotent) rather than quadratic power (periodic with period two) that returns pos-one (for the three positive triple cross quaternions).

The double crosses place scalar exponential attenuation factors on pairs of the diagonal, whilst the mixed triple cross quaternions exponentials place scalar exponential attenuation factors on a single components of the diagonal (corresponding to the basis-hat to which they correspond as an idempotent operator). For example, that for k -hat, **expikj** = **cisikj** [which is a power series in $\hat{i} \times \hat{k} \times \hat{j} \times$] has only the non-null term, that in the k -hat direction, and this is the identity component for k -hat. Expanding **expikj** in a power series

$$\exp(\theta \hat{i} \times \hat{k} \times \hat{j} \times) = 1 + \theta(\hat{i} \times \hat{k} \times \hat{j} \times) + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{k} \times \hat{j} \times)^2 + \frac{1}{3!} \theta^3 (\hat{i} \times \hat{k} \times \hat{j} \times)^3 + \dots$$

- column basis i -hat direction is the i -hat unit vector

$$\exp(\theta \hat{i} \times \hat{k} \times \hat{j} \times) \hat{i} = \hat{i} + \theta(\hat{i} \times \hat{k} \times \hat{j} \times) \hat{i} + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{k} \times \hat{j} \times)^2 \hat{i} + \dots = \hat{i} + 0\hat{i} + 0\hat{i} + \dots = \hat{i}$$

- column basis j -hat direction is the j -hat unit vector

$$\exp(\theta \hat{i} \times \hat{k} \times \hat{j} \times) \hat{j} = \hat{j} + \theta(\hat{i} \times \hat{k} \times \hat{j} \times) \hat{j} + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{k} \times \hat{j} \times)^2 \hat{j} + \dots = \hat{j} + 0\hat{j} + 0\hat{j} + \dots = \hat{j}$$

- column basis k -hat direction returns scalar exponential on k -hat

$$\begin{aligned} \exp(\theta \hat{i} \times \hat{k} \times \hat{j} \times) \hat{k} &= \hat{k} + \theta(\hat{i} \times \hat{k} \times \hat{j} \times) \hat{k} + \frac{1}{2!} \theta^2 (\hat{i} \times \hat{k} \times \hat{j} \times)^2 \hat{k} + \dots \\ &= \hat{k} + \theta \hat{k} + \frac{1}{2!} \theta^2 \hat{k} + \dots \\ &= \left(1 + \theta + \frac{1}{2!} \theta^2 + \dots\right) \hat{k} \\ &= \hat{k} \left(1 + \theta + \frac{1}{2!} \theta^2 + \dots\right) \\ &= \hat{k} \exp(\theta) \end{aligned}$$

The diadic expansion of the three components, mapping the components to columns, results in

$$\exp(\theta \hat{i} \times \hat{k} \times \hat{j} \times) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^\theta \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Hence the exponentiation of $\mathbf{i} \times \mathbf{k} \times \mathbf{j}$ for **expikj** is the scalar exponential placed on the k -hat component of the diagonal of the identity, that is, on the kk' position of I .

The expansions to matrix base operators for the other two positive triple cross quaternions are similar. The three basis exponentials for the positive triple cross quaternions as a set are

$$\begin{aligned} \mathbf{expikj} \theta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^\theta \end{pmatrix} & \mathbf{expkji} \theta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathbf{expjik} \theta &= \begin{pmatrix} e^\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which magnify on the idempotent entry of the triple-cross. The basis exponentials of the negative triple cross quaternions are

$$\begin{aligned} \mathbf{expjki} \theta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix} & \mathbf{expijk} \theta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \mathbf{expkij} \theta &= \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which attenuate rather than magnify on the exponential entry of the diagonal.

The sine and cosine basis functions (of $\mathbf{k} \times \mathbf{j} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{i} \times \mathbf{k}$, $\mathbf{i} \times \mathbf{k} \times \mathbf{j}$) map hyperbolics to a single axis (rather than two axes for the double crosses of $(\mathbf{k} \times)^2 = -1$, $(\mathbf{j} \times)^2 = -1$, and $(\mathbf{i} \times)^2 = -1$). Because cosine is an even function, the positive triple cross quaternion cosines (**cosijk**, **cosjki**, **coskij**) are the same as the negative triple cross quaternion cosines (**coskji**, **cosjik**, **cosikj**)

$$\mathbf{cosikj}\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix}$$

$$\mathbf{coskji}\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{cosjik}\theta = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{cosjki}\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix}$$

$$\mathbf{cosijk}\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{coskij}\theta = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Because sine is an odd function, the **positive triple cross quaternion** sines ($\mathbf{sinijk}$, $\mathbf{sinjki}$, $\mathbf{sinkij}$) differ in sign to the **negative triple cross quaternion** sines ($\mathbf{sinkji}$, $\mathbf{sinjik}$, $\mathbf{sinikj}$)

$$\mathbf{sinikj}\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix}$$

$$\mathbf{sinkji}\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

$$\mathbf{sinjik}\theta = \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

$$\mathbf{sinjki}\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & -\sinh\theta \end{pmatrix}$$

$$\mathbf{sinijk}\theta = \begin{pmatrix} \theta & 0 & 0 \\ 0 & -\sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

$$\mathbf{sinkij}\theta = \begin{pmatrix} -\sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

Since any basis exponential has a Euler-type form, these similarly can also be expressed in Euler form ($\mathbf{cisijk}$, $\mathbf{cisjki}$, $\mathbf{ciskij}$) and ($\mathbf{ciskji}$, $\mathbf{cisjik}$, $\mathbf{cisikj}$). The Euler forms are

$$\mathbf{cisikj}\theta = \mathbf{cosikj}\theta + (\mathbf{cosikj}\theta)\hat{i} \times \hat{k} \times \hat{j} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} + \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^\theta \end{pmatrix} = \mathbf{expikj}\theta$$

$$\mathbf{ciskji}\theta = \mathbf{coskji}\theta + (\mathbf{coskji}\theta)\hat{j} \times \hat{k} \times \hat{i} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{expkji}\theta$$

$$\mathbf{cisjik}\theta = \mathbf{cosjik}\theta + (\mathbf{cosjik}\theta)\hat{j} \times \hat{i} \times \hat{k} \times = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e^\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{expjik}\theta$$

$$\mathbf{cisjki}\theta = \mathbf{cosjki}\theta + (\mathbf{cosjki}\theta)\hat{j} \times \hat{k} \times \hat{i} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh\theta \end{pmatrix} - \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\theta} \end{pmatrix} = \mathbf{expjki}\theta$$

$$\mathbf{cisijk}\theta = \mathbf{cosijk}\theta + (\mathbf{cosijk}\theta)\hat{i} \times \hat{j} \times \hat{k} \times = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta & 0 & 0 \\ 0 & \sinh\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{expijk}\theta$$

$$\mathbf{ciskij}\theta = \mathbf{coskij}\theta + (\mathbf{coskij}\theta)\hat{k} \times \hat{i} \times \hat{j} \times = \begin{pmatrix} \cosh\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \sinh\theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{-\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{expkij}\theta$$

They definitely look nice, although unlike those for **expk**, **expj**, and **expi** which are known to be extraordinarily useful (as rotation matrices and complex exponentials), I am not sure how useful these will be. I think it is best that they are fully elucidated anyway which is what I have decided to do here. Since any basis exponential has a **Euler-type form**, there are **de-Moivre powers** for any basis exponential and it is self-evident here that **de-Moivre powers** can be taken for ($\mathbf{expijk} \theta^n = \mathbf{cisijk} n\theta$ and ($\mathbf{expjki} \theta^n = \mathbf{cisjki} n\theta$ and ($\mathbf{expkji} \theta^n = \mathbf{ciskji} n\theta$ for the **positive triple cross quaternions** and similarly ($\mathbf{expkij} \theta^n = \mathbf{ciskij} n\theta$ and ($\mathbf{expjik} \theta^n = \mathbf{cisjik} n\theta$ and ($\mathbf{expikj} \theta^n = \mathbf{cisikj} n\theta$ for the **negative triple cross quaternions**

$$(\mathbf{expikj}\theta)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{n\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh n\theta \end{pmatrix} + \begin{pmatrix} n\theta & 0 & 0 \\ 0 & n\theta & 0 \\ 0 & 0 & \sinh n\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{cosikjn}\theta + (\mathbf{cosikjn}\theta)\hat{i} \times \hat{k} \times \hat{j} \times = \mathbf{cisikj}(n\theta)$$

$$(\mathbf{expkji} \theta)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{n\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} n\theta & 0 & 0 \\ 0 & \sinh n\theta & 0 \\ 0 & 0 & n\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{coskjin}\theta + (\mathbf{sinkjin}\theta)\hat{j} \times \hat{k} \times \hat{i} \times = \mathbf{ciskji}(n\theta)$$

$$(\mathbf{expjik} \theta)^n = \begin{pmatrix} e^{n\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh n\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \sinh n\theta & 0 & 0 \\ 0 & n\theta & 0 \\ 0 & 0 & n\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{cosjikn}\theta + (\mathbf{sinjikn}\theta)\hat{j} \times \hat{i} \times \hat{k} \times = \mathbf{cisjik}(n\theta)$$

$$(\mathbf{expkji} \theta)^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-n\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cosh n\theta \end{pmatrix} - \begin{pmatrix} n\theta & 0 & 0 \\ 0 & n\theta & 0 \\ 0 & 0 & \sinh n\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{cosjkin}\theta + (\mathbf{sinjkin}\theta)\hat{j} \times \hat{k} \times \hat{i} \times = \mathbf{cisjki}(n\theta)$$

$$(\mathbf{expijk} \theta)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-n\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh n\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} n\theta & 0 & 0 \\ 0 & \sinh n\theta & 0 \\ 0 & 0 & n\theta \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{cosijkn}\theta + (\mathbf{sinijkn}\theta)\hat{i} \times \hat{j} \times \hat{k} \times = \mathbf{cisijk}(n\theta)$$

$$(\mathbf{expkij} \theta)^n = \begin{pmatrix} e^{-n\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh n\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \sinh n\theta & 0 & 0 \\ 0 & n\theta & 0 \\ 0 & 0 & n\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{coskijn}\theta + (\mathbf{coskijn}\theta)\hat{k} \times \hat{i} \times \hat{j} \times = \mathbf{ciskij}(n\theta)$$

Rather than rotating on a single axis as for rotations around an Argand diagram and plane kinematics with $\mathbf{k} \times$ and $\mathbf{expk}$, it is interesting how crossing from one axis to another introduces extensive nilpotence from the nullity in $\mathbf{k} \times$, $\mathbf{j} \times$, and $\mathbf{i} \times$. Although the **quaternion nilpotent triple crosses** seem trivial, they may be essential in the future for full dynamics in R^3 .